

Stability of periodic waves in the defocusing cubic NLS equation

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Higher-order conserved quantities associated with nonlinear integrable equations have been used in analysis of orbital stability of nonlinear waves:

- ▶ Orbital stability of n -solitons in $H^n(\mathbb{R})$ was proved by Sachs - Maddocks (1993) for KdV and by Kapitula (2006) for NLS.
- ▶ Orbital stability of breathers in $H^2(\mathbb{R})$ was proved by Alejo–Munoz (2013) for the modified KdV equation.
- ▶ Orbital stability of Dirac solitons in $H^1(\mathbb{R})$ was proved by P–Shimabukuro (2014) for the massive Thirring model.
- ▶ Orbital stability of periodic waves with respect to subharmonic perturbations was considered by Deconinck and collaborators for KdV (2010) and defocusing NLS (2011).

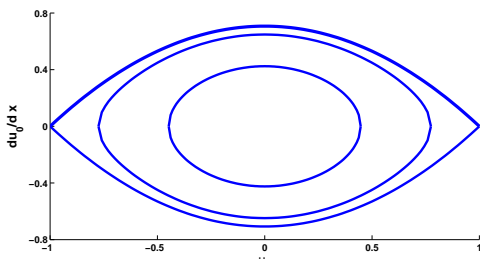
The defocusing NLS equation

$$i\psi_t + \psi_{xx} - |\psi|^2\psi = 0, \quad \psi = \psi(x, t).$$

Periodic waves exist in the form $\psi(x, t) = u_0(x)e^{-it}$, where

$$\frac{d^2 u_0}{dx^2} + (1 - |u_0|^2)u_0 = 0,$$

in fact, in the explicit form $u_0(x) = \sqrt{1 - \mathcal{E}} \operatorname{sn} \left(x \frac{\sqrt{1+\mathcal{E}}}{\sqrt{2}}; \sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}} \right)$,
where $\mathcal{E} \in (0, 1)$ is a free parameter.



Periodic waves are critical points of the energy

$$E(\psi) = \int \left[|\psi_x|^2 + \frac{1}{2}(1 - |\psi|^2)^2 \right] dx$$

However, they are not minimizers of the energy even in the space of periodic perturbations.

- ▶ Gallay–Haragus (2007) proved orbital stability w.r.t. perturbations with the same period as $|u_0|$ by using constraints from lower-order conserved quantities:

$$Q(\psi) = \int |\psi|^2 dx, \quad M(\psi) = \frac{i}{2} \int (\bar{\psi}\psi_x - \psi\bar{\psi}_x) dx.$$

The periodic waves are constrained minimizers of the energy.

- ▶ Bottman–Deconinck–Nivala (2011) proved spectral stability of periodic waves in the Floquet–Bloch theory and used the higher-order conserved quantity

$$R(\psi) = \int \left[|\psi_{xx}|^2 + 3|\psi|^2|\psi_x|^2 + \frac{1}{2}(\bar{\psi}\psi_x + \psi\bar{\psi}_x)^2 + \frac{1}{2}|\psi|^6 \right] dx,$$

to show that the periodic waves are critical points of the higher-order energy $S := R - \frac{1}{2}(3 - \mathcal{E}^2)Q$.

- ▶ Periodic waves are not minimizers of neither E nor S . Nevertheless, the energy functional $\Lambda_c := S - cE$ is claimed to be positively definite at u_0 for some values of parameter c .
- ▶ Motivation for our work is to understand the constraints on c and to prove the claim by using rigorous PDE analysis.

Lemma

There exists $\mathcal{E}_0 \in (0, 1)$ s.t. for all $\mathcal{E} \in (\mathcal{E}_0, 1)$, there exist values c_- and c_+ in the range $1 < c_- < 2 < c_+ < 3$ s.t. the second variation of Λ_c at the periodic wave u_0 is nonnegative for perturbations in $H^2(\mathbb{R})$ if $c \in (c_-, c_+)$. Moreover,

$$c_{\pm} = 2 \pm \sqrt{2(1 - \mathcal{E})} + \mathcal{O}(1 - \mathcal{E}) \quad \text{as } \mathcal{E} \rightarrow 1.$$

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Lemma

For all $\mathcal{E} \in (0, 1)$, the second variation of Λ_c at the periodic wave u_0 is nonnegative for perturbations in $H^2(\mathbb{R})$ only if $c \in [c_-, c_+]$ with

$$c_{\pm} := 2 \pm \frac{2\kappa}{1 + \kappa^2}, \quad \kappa = \sqrt{\frac{1 - \mathcal{E}}{1 + \mathcal{E}}}.$$

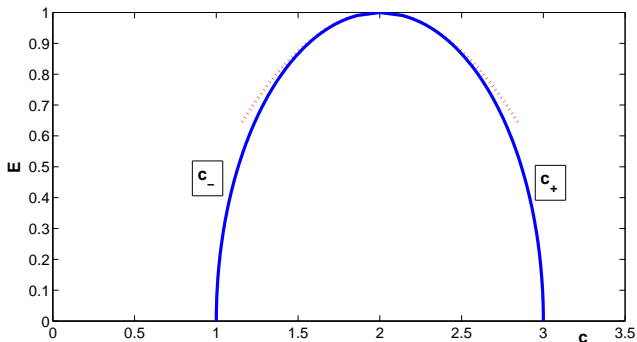


Figure : (\mathcal{E}, c) -plane for positivity of the second variation of Λ_c .

Using the decomposition $\psi = u_0 + u + iv$ with real-valued perturbation functions u and v , we can write

$$\Lambda_c(\psi) - \Lambda_c(u_0) = \langle K_+(c)u, u \rangle_{L^2} + \langle K_-(c)v, v \rangle_{L^2} + \text{cubic terms}$$

where

$$K_+(c)\partial_x u_0 = 0 \quad \text{and} \quad K_-(c)u_0 = 0.$$

Lemma

Fix $c = 2$. For any $\mathcal{E} \in (0, 1)$ and any $v \in H^2(\mathbb{R})$, we have

$$\langle K_-(2)v, v \rangle_{L^2} = \|v_{xx} + (1 - u_0^2)v\|_{L^2}^2 + \|u_0 v_x - u_0' v\|_{L^2}^2.$$

Moreover, $\langle K_-(2)v, v \rangle_{L^2} = 0$ if and only if $v = Cu_0$.

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$$\langle K_+(2)u, u \rangle_{L^2} \geq 0.$$

Moreover, $\langle K_+(2)u, u \rangle_{L^2} = 0$ if and only if $u = C\partial_x u_0$.

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Theorem

Assume that $\psi_0 \in H_{\text{per}}^2(0, T)$, where T is a multiple of the period of u_0 , and consider the global-in-time solution ψ to the cubic NLS equation with initial data ψ_0 . For any $\epsilon > 0$, there is $\delta > 0$ s.t. if

$$\|\psi_0 - u_0\|_{H_{\text{per}}^2(0, T)} \leq \delta,$$

then, for any $t \in \mathbb{R}$, there exist numbers $\xi(t)$ and $\theta(t)$ such that

$$\|e^{i(t+\theta(t))}\psi(\cdot + \xi(t), t) - u_0\|_{H_{\text{per}}^2(0, T)} \leq \epsilon.$$

Let us give a simple argument why $\Lambda_c = S - cE$ can be positive definite at the periodic wave u_0 . For $\mathcal{E} = 1$, we have $u_0 = 0$, and

$$\langle K_{\pm}(c)u, u \rangle_{L^2} = \int_{\mathbb{R}} [u_{xx}^2 - cu_x^2 + (c-1)u^2] dx.$$

Integration by parts yields

$$\langle K_{\pm}(c)u, u \rangle_{L^2} = \int \left(u_{xx} + \frac{c}{2}u \right)^2 dx - \left(1 - \frac{c}{2} \right)^2 \int u^2 dx,$$

which is non-negative if $c = 2$.

However, the case $\mathcal{E} = 1$ is degenerate, hence perturbation arguments are needed to unfold the degeneracy for $\mathcal{E} < 1$.

To run perturbation arguments, we normalize the period:

$$u_0(x) = U(z), \quad z = \ell x, \quad U(z + 2\pi) = U(z).$$

The function $U(z)$ satisfies

$$\ell^2 \frac{d^2 U}{dz^2} + U - U^3 = 0 \quad \Rightarrow \quad \ell^2 \left(\frac{dU}{dz} \right)^2 = \frac{1}{2} [(1 - U^2)^2 - \mathcal{E}^2].$$

Lemma

The map $(0, 1) \ni \mathcal{E} \mapsto (\ell, U) \in \mathbb{R} \times H_{\text{per}}^2(0, 2\pi)$ can be uniquely parameterized by the small parameter a as $\mathcal{E} \rightarrow 1$ such that

$$\mathcal{E} = 1 - a^2 + \mathcal{O}(a^4), \quad \ell^2 = 1 - \frac{3}{4}a^2 + \mathcal{O}(a^4), \quad U(z) = a \cos(z) + \mathcal{O}(a^3).$$

Operators $K_{\pm}(c)$ from $H^4(\mathbb{R})$ to $L^2(\mathbb{R})$ have 2π -periodic coefficients in variable z . By Floquet–Bloch theory, λ belongs to the purely continuous spectrum of $K_{\pm}(c)$ in $L^2(\mathbb{R})$ if there exists a bounded solution of the spectral problem

$$K_{\pm}(c)u(z, k) = \lambda u(z, k), \quad u(z, k) := e^{ikz} w(z, k),$$

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where w is 2π -periodic in z and 1-periodic in k .

If $a = 0$, then the Floquet–Bloch spectrum of

$$K_{\pm}(c) = \partial_z^4 + c\partial_z^2 + (c - 1)$$

is found from the Fourier series:

$$\lambda_n^0(k) = (n + k)^4 - c(n + k)^2 + c - 1, \quad n \in \mathbb{Z}.$$

If $c = 2$, then $\lambda_n^0(k) = 0$ if and only if $k = 0$ and $n = \pm 1$.

Lemma

If $a > 0$ is sufficiently small and $c \in (c_-, c_+)$, where $c_{\pm} = 2 \pm \sqrt{2}a + \mathcal{O}(a^2)$, the operator $K_{\pm}(c)$ has exactly one Floquet–Bloch band denoted by $\text{range}(\lambda_{-1}^{\pm}(k))$ that touches the origin at $k = 0$, while all other bands are strictly positive.

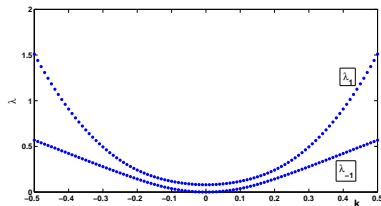
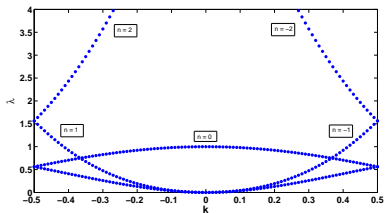


Figure : Spectral bands for $c = 2$ and $a = 0$ (left). Degenerate spectral bands for $c = 2$ and $a = 0.2$ (right).

Consider the operator $K_-(c)$ and the spectral band $\lambda_{-1}^-(k)$ that touches zero at $k = 0$ because $K_-(c)u_0 = 0$.

Lemma

Fix $\mathcal{E} \in (0, 1)$ and assume that u_0 is the only 2π -periodic solution of $K_-(c)w = 0$. Then, λ_{-1}^- is C^2 near $k = 0$ with $\lambda_{-1}^-(0) = \lambda_{-1}'(0) = 0$, and

$$\lambda_{-1}^{-''}(0) = \frac{2\ell^2 \kappa^2 K(\kappa) (4\kappa^2 - (c-2)^2 (1+\kappa^2)^2)}{(1+\kappa^2)(K(\kappa) - E(\kappa)) \left(2\kappa^2 + (c-2)(1+\kappa^2) \left(1 - \frac{E(\kappa)}{K(\kappa)} \right) \right)}.$$

Corollary

$\lambda_{-1}^{-''}(0) > 0$ if $c \in (c_-, c_+)$, where c_{\pm} are defined by

$$c_{\pm} := 2 \pm \frac{2\kappa}{1+\kappa^2}, \quad \kappa = \sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}}.$$

The perturbative arguments only apply to the periodic waves of small amplitudes (when \mathcal{E} is close to 1). The question is how to continue these arguments to large-amplitude waves.

Lemma

Fix $c = 2$. For any $\mathcal{E} \in [0, 1]$ and any $v \in H^2(\mathbb{R})$, we have

$$\langle K_-(2)v, v \rangle_{L^2} = \|L_- v\|_{L^2}^2 + \|u_0 v_x - u_0' v\|_{L^2}^2,$$

where $L_- := -\partial_x^2 + u_0^2(x) - 1$. Note that $L_- u_0 = K_-(c)u_0 = 0$.

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$$\langle K_-(2)v, v \rangle_{L^2} = \|L_- v\|_{L^2}^2 + \|u_0 v_x - u'_0 v\|_{L^2}^2,$$

where $L_- := -\partial_x^2 + u_0^2(x) - 1$. Note that $L_- u_0 = K_-(c)u_0 = 0$.

When we apply the same idea to $K_+(c)$ with $c = 2$, we only obtain

$$\langle K_+(2)u, u \rangle_{L^2} = \|L_+ u\|_{L^2}^2 - \int [u_0^2 u_x^2 - 3u_0^2 u^2 + 5u_0^4 u^2] dx,$$

where $L_+ := -\partial_x^2 + 3u_0^2(x) - 1$. Note that $L_+ u'_0 = K_+(c)u'_0 = 0$.

In order to obtain the positivity of $K_+(2)$, we apply the following chain of arguments.

1. Operators L_{\pm} and $K_{\pm}(c)$ commute for any $c \in \mathbb{R}$ as follows:

$$L_- K_+(c) = K_-(c) L_+, \quad L_+ K_-(c) = K_+(c) L_-.$$

This follows from commutability of the evolution flows of the integrable NLS hierarchy.

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This follows from commutability of the evolution flows of the integrable NLS hierarchy.

2. Bounded solutions of L_{\pm} are given as follows:

- ▶ If $u \in L^\infty(\mathbb{R}) \cap H_{loc}^2(\mathbb{R})$ satisfies $L_+ u = 0$, then $u = C \partial_x u_0$.
- ▶ If $v \in L^\infty(\mathbb{R}) \cap H_{loc}^2(\mathbb{R})$ satisfies $L_- v = 0$, then $v = C u_0$.

This follows from analysis of Schrödinger operators L_{\pm} with periodic coefficients.

3 If $v \in L^\infty(\mathbb{R}) \cap H_{\text{loc}}^4(\mathbb{R})$ satisfies $K_-(2)v = 0$, then $v = Cu_0$.

If $K_-(2)v = 0$, then for every $N \in \mathbb{N}$ including $N \rightarrow \infty$,

$$\begin{aligned} 0 &= \frac{1}{N} \int_0^{NT_0} (|L_- v|^2 + |u_0 v_x - u'_0 v|^2) dx + \frac{1}{N} \text{b.v.} \Big|_{x=0}^{x=NT_0} \\ &= \int_0^{T_0} (|L_- v|^2 + |u_0 v_x - u'_0 v|^2) dx + \frac{1}{N} \text{b.v.} \Big|_{x=0}^{x=NT_0}. \end{aligned}$$

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4 If $u \in L^\infty(\mathbb{R}) \cap H_{loc}^4(\mathbb{R})$ solves $K_+(2)u = 0$, then $u = C\partial_x u_0$.

Indeed, $K_-(2)L_+ u = L_- K_+(2)u = 0$ implies $L_+ u = Bu_0$.

Then, $u = BU + C\partial_x u_0$, where $U = L_+^{-1}u_0$ exists. However, $K_+(2)u = BK_+(2)U$ with $K_+(2)U \neq 0$, hence $B = 0$.

- 5 For every $\mathcal{E} \in (0, 1)$, all spectral bands of the operator $K_+(2)$ cannot touch $\lambda = 0$ except for the lowest band, which touches $\lambda = 0$ exactly at $k = 0$, with $u = C\partial_x u_0$.

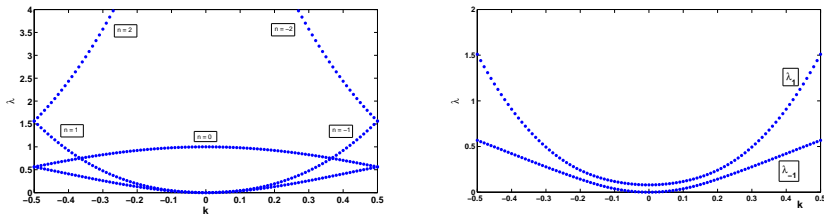


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Let T be a multiple of the period T_0 of the periodic wave u_0 . For the unique global solution $\psi(x, t)$ of the cubic defocusing NLS equation in $H_{\text{per}}^2(0, T)$, we write

$$e^{i\theta(t)+it}\psi(x + \xi(t), t) = u_0(x) + u(x, t) + iv(x, t),$$

where real u and v satisfy the orthogonality conditions

$$\langle \partial_x u_0, u(\cdot, t) \rangle_{L_{\text{per}}^2} = 0, \quad \langle u_0, v(\cdot, t) \rangle_{L_{\text{per}}^2} = 0, \quad t \in \mathbb{R}.$$

The functions $\theta(t)$ and $\xi(t)$ are uniquely determined by the orthogonality conditions if $\|u\|_{H_{\text{per}}^2}$ and $\|v\|_{H_{\text{per}}^2}$ are small. From coercivity of the quadratic forms, we obtain

$$\langle K_+(2)u, u \rangle_{L_{\text{per}}^2} \geq C_+ \|u\|_{H_{\text{per}}^2}^2, \quad \langle K_-(2)v, v \rangle_{L_{\text{per}}^2} \geq C_- \|v\|_{H_{\text{per}}^2}^2.$$

The rest follows from the energy conservation

$$\begin{aligned}\Delta\Lambda_{c=2} &:= \Lambda_{c=2}(\psi(\cdot, t)) - \Lambda_{c=2}(u_0) \\ &= \Lambda_{c=2}(e^{i\theta(t)+it}\psi(\cdot + \xi(t), t)) - \Lambda_{c=2}(u_0) \\ &= \langle K_+(2)u, u \rangle_{L^2_{\text{per}}} + \langle K_-(2)v, v \rangle_{L^2_{\text{per}}} + N(u, v).\end{aligned}$$

If $\|\psi_0 - u_0\|_{H^2_{\text{per}}(0, T)} \leq \delta$, then $|\Delta\Lambda_{c=2}| \leq C\delta^2$. From coercivity of the quadratic part and smallness of the cubic part $N(u, v)$, we obtain for all $t \in \mathbb{R}$:

$$\|e^{i\theta(t)+it}\psi(\cdot + \xi(t), t) - u_0\|_{H^2_{\text{per}}} \leq C|\Delta\Lambda_{c=2}|^{1/2} \leq C\delta =: \epsilon.$$

Open questions:

- ▶ Orbital stability of periodic waves with respect to perturbations in $H^2(\mathbb{R})$.
- ▶ Extension of this analysis for non-real periodic waves with a nontrivial phase.
- ▶ Extension of this analysis to other integrable equations (e.g. the Korteweg–de Vries equation).

Reference:

- ▶ Th. Gallay and D.P., J. Diff. Eqs. (2014), accepted.