Ground and excited states in a parabolic trap

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Kyushu University, Fukuoka, July 16, 2010

References:

M. Coles, D.P., P. Kevrekidis, Nonlinearity **23**, 1753–1770 (2010)
P., Nonlinear Analysis, accepted (2010).

Introduction

Density waves in cigar–shaped Bose–Einstein condensates with repulsive inter-atomic interactions and a harmonic potential are modeled by the Gross-Pitaevskii equation

$$iv_\tau = -\frac{1}{2}v_{\xi\xi} + \frac{1}{2}\xi^2 v + |v|^2 v - \mu v,$$

where μ is the chemical potential.

Using the scaling transformation,

$$v(\xi, t) = \mu^{1/2} u(x, t), \quad \xi = (2\mu)^{1/2} x, \quad \tau = 2t,$$

the Gross–Pitaevskii equation is transformed to the semi-classical form

$$i\,\varepsilon\, u_t + \varepsilon^2\, u_{xx} + (1 - x^2 - |u|^2)u = 0,$$

where $\varepsilon = (2\mu)^{-1}$ is a small parameter.

Ground state

Limit $\mu \to \infty$ or $\varepsilon \to 0$ is referred to as the semi-classical or Thomas–Fermi limit. Physically, it is the limit of large density of the atomic cloud.

Let η_ε be the positive solution of the stationary problem (ground state)

$$\varepsilon^2 \eta_\varepsilon''(x) + (1 - x^2 - \eta_\varepsilon^2(x))\eta_\varepsilon(x) = 0, \quad x \in \mathbb{R}.$$

Theorem (Ignat & Milot, JFA (2006))

For sufficiently small $\varepsilon > 0$, there exists a global minimizer of the Gross–Pitaevskii energy

$$E_\varepsilon(u) = \int_{\mathbb{R}} \left(\frac{1}{2} \varepsilon^2 |u_x|^2 + \frac{1}{2}(x^2 - 1)|u|^2 + \frac{1}{4}|u|^4 \right) dx$$

in the energy space

$$\mathcal{H}_1 = \left\{ u \in H^1(\mathbb{R}) : \ \ xu \in L^2(\mathbb{R}) \right\}.$$

Ground state in the asymptotic theory

For small $\varepsilon > 0$, the ground state $\eta_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ decays to zero as $|x| \to \infty$ faster than any exponential function and satisfies

$$\eta_0(x) := \lim_{\varepsilon \to 0} \eta_\varepsilon(x) = \begin{cases} (1 - x^2)^{1/2}, & \text{for } |x| < 1, \\ 0, & \text{for } |x| > 1, \end{cases}$$

- For any compact subset $K \subset (-1, 1)$, there is $C_K > 0$ such that

$$\|\eta_\varepsilon - \eta_0\|_{C^1(K)} \leq C_K \, \varepsilon^2 .$$

- There is $C > 0$ such that

$$\|\eta_\varepsilon - \eta_0\|_{L^\infty} \leq C \, \varepsilon^{1/3}, \quad \|\eta_\varepsilon'\|_{L^\infty} \leq C \, \varepsilon^{-1/3} .$$

- There is $C > 0$ such that

$$C \, \varepsilon^{1/3} \leq \eta_\varepsilon(x) \leq 1, \;\; |x| \leq 1, \quad 0 \leq \eta_\varepsilon(x) \leq C \, \varepsilon^{1/3} \exp\left(\frac{1 - x^2}{4 \, \varepsilon^{2/3}}\right) \;\; |x| \geq 1.$$

Gallo & P., Asymptotic Analysis (2010)

Excited states in the asymptotic theory

Let u_ε be the non-positive solution of the stationary problem (an excited state)

$$\varepsilon^2 u_\varepsilon''(x) + (1 - x^2 - u_\varepsilon^2(x))u_\varepsilon(x) = 0, \quad x \in \mathbb{R}.$$

The excited states are classified by the number m of zeros of $u_\varepsilon(x)$ on $\mathbb{R}$.

The product representation

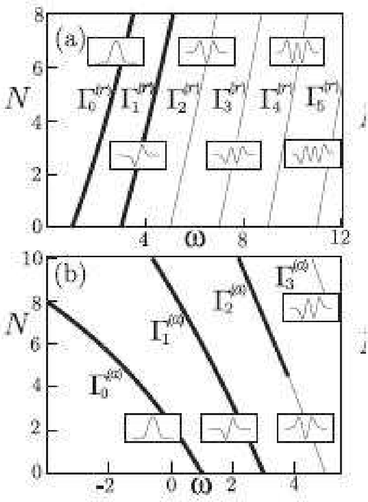
$$u(x,t) = \eta_\varepsilon(x) v(x,t)$$

brings the Gross–Pitaevskii equation to the equivalent form

$$i\,\varepsilon\,\eta_\varepsilon^2 v_t + \varepsilon^2 \left(\eta_\varepsilon^2 v_x\right)_x + \eta_\varepsilon^4 (1 - |v|^2) v = 0,$$

where $\lim_{x \to \pm\infty} |v(x)| = 1$.

Stability of the *m*-th excited state



Zezulin, Alfimov, Konotop, & Perez–Garcia, PRA (2008)

Main objectives

- Justify the asymptotic bounds on the ground state η_ε
- Study variational approximations of the *m*-th excited state
- Justify the variational results using rigorous methods
- Study distribution of eigenfrequencies of the ground and excited states
- Extend the results to vortices in two and three dimensions.

Asymptotic construction of the ground state

Let

$$\eta_\varepsilon(x) = \varepsilon^{1/3}\, \nu_\varepsilon(y), \quad y = \frac{1 - x^2}{\varepsilon^{2/3}}$$

and write an equation on $\eta_\varepsilon(y)$:

$$4(1 - \varepsilon^{2/3}\, y)\nu_\varepsilon''(y) - 2\,\varepsilon^{2/3}\, \nu_\varepsilon'(y) + y\nu_\varepsilon(y) - \nu_\varepsilon^3(y) = 0, \quad y \in J_\varepsilon,$$

where

$$J_\varepsilon := (-\infty, \varepsilon^{-2/3})$$

and $\nu_\varepsilon(y)$ decays to zero as $y \to -\infty$ and satisfies the Neumann boundary condition at $\varepsilon^{-2/3}$:

$$\eta_\varepsilon'(0) = 0 \quad \Longleftrightarrow \quad \lim_{y \uparrow \varepsilon^{-2/3}} \sqrt{1 - \varepsilon^{2/3}\, y}\,\nu_\varepsilon'(y) = 0.$$

Asymptotic construction of the ground state

Fix $N \geq 0$ and look for solutions in the form

$$\nu_\varepsilon(y) = \sum_{n=0}^{N} \varepsilon^{2n/3}\, \nu_n(y) + \varepsilon^{2(N+1)/3}\, R_{N,\varepsilon}(y), \quad y \in J_\varepsilon,$$

- ν_0 solves the Painlevé-II equation

$$4\nu_0''(y) + y\nu_0(y) - \nu_0^3(y) = 0, \quad y \in \mathbb{R},$$

- for $1 \leq n \leq N$, ν_n solves

$$M_0\nu_n := -4\nu_n''(y) + \left(3\nu_0^2(y) - y\right)\nu_n(y) = F_n(y), \quad y \in \mathbb{R},$$

- $R_{N,\varepsilon}$ solves

$$-4(1-\varepsilon^{2/3}y)R_{N,\varepsilon}'' + 2\,\varepsilon^{2/3}\, R_{N,\varepsilon}' + \left(3\nu_0^2(y) - y\right)R_{N,\varepsilon} = F_{N,\varepsilon}(y, R_{N,\varepsilon}), \quad y \in J_\varepsilon,$$

Remark: $\nu_n(y)$ does not depend on ε and is defined on $\mathbb{R}$.

Main result

Theorem

Let ν_0 be the unique solution of the Painlevé II equation such that

$$\nu_0(y) \sim y^{1/2} \quad \text{as} \quad y \to +\infty \quad \text{and} \quad \nu_0(y) \to 0 \quad \text{as} \quad y \to -\infty.$$

For $n \geq 1$, ν_n is the unique solution of the linearized Painlevé equation in $\mathcal{C}^2(\mathbb{R}) \cap L^2(\mathbb{R})$. For every $N \geq 0$, there exists $\varepsilon_N > 0$ and $C_N > 0$ such that for every $0 < \varepsilon < \varepsilon_N$, there is

$$R_{N,\varepsilon} \in L^\infty(J_\varepsilon), \quad \text{with} \quad \|R_{N,\varepsilon}\|_{L^\infty(J_\varepsilon)} \leq C_N, \quad \lim_{y \to -\infty} R_{N,\varepsilon}(y) = 0,$$

such that for every $x \in \mathbb{R}$,

$$\eta_\varepsilon(x) = \varepsilon^{1/3} \sum_{n=0}^{N} \varepsilon^{2n/3} \nu_n \left(\frac{1 - x^2}{\varepsilon^{2/3}} \right) + \varepsilon^{2N/3+1} R_{N,\varepsilon} \left(\frac{1 - x^2}{\varepsilon^{2/3}} \right).$$

Step I: Hasting-McLeod solution

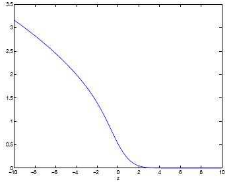
The Painlevé-II equation

$$4\nu''(y) + y\nu(y) - \nu^3(y) = 0, \quad y \in \mathbb{R},$$

admits a unique solution $\nu_0 \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\nu_0(y) = \frac{1}{2\sqrt{\pi}}(-2y)^{-1/4} e^{-\frac{2}{3}(-2y)^{3/2}} \left(1 + \mathcal{O}(|y|^{-3/4})\right) \underset{y \to -\infty}{\approx} 0,$$

$$\nu_0(y) \underset{y \to +\infty}{\approx} y^{1/2} \sum_{n=0}^{\infty} \frac{b_n}{(2y)^{3n/2}}.$$



12. Hastings-McLeod solution of the Painlevé II equation.

Fokas, Its, Kapaev, Novokshenov, AMS Monographs (2006)

Step II: Linearized Painlevé-II equation

Let us consider the operator M_0 on $L^2(\mathbb{R})$, defined by

$$M_0 := -4\partial_y^2 + W_0(y), \quad W_0(y) = 3\nu_0^2(y) - y.$$

From the asymptotic behaviors of $\nu_0(y)$ as $y \to \pm\infty$, we infer that

$$W_0(y) \sim 2y \quad \text{as} \quad y \to +\infty \quad \text{and} \quad W_0(y) \sim -y \quad \text{as} \quad y \to -\infty.$$

Moreover, we prove that

$$\inf_{y \in \mathbb{R}} W_0(y) > 0$$

and $W_0(y)$ has the only extremum at the global minimum near $y = 0$.

For any $n \in \{1, 2, ..., N\}$, corrections $\nu_n \in \mathcal{C}^2(\mathbb{R}) \cap L^2(\mathbb{R})$ are found from the inhomogeneous equations $M_0 \nu_n = f_n$ such that

$$\nu_n(y) \underset{y \to +\infty}{\approx} y^{-5/2-2n} \sum_{m=0}^{\infty} g_{n,m} y^{-3m/2}, \quad \nu_n(y) \underset{y \to -\infty}{\approx} 0.$$

Step III: Remainder term

The remainder term satisfies

$$T^{\varepsilon} R_{N,\varepsilon}(y) = \frac{F_{N,\varepsilon}(y, R_{N,\varepsilon})}{\sqrt{1-\varepsilon^{2/3} y}}, \quad y \in J_{\varepsilon},$$

where

$$T^{\varepsilon} = -4\partial_y \sqrt{1-\varepsilon^{2/3} y}\,\partial_y + \frac{W_0(y)}{\sqrt{1-\varepsilon^{2/3} y}}$$

and $F_{N,\varepsilon}(y,R) = F_{N,0}(y) + G_{N,\varepsilon}(y,R)$ with

$$\|F_{N,0}\|_{L^2_\varepsilon} \lesssim 1, \quad \|G_{N,\varepsilon}\|_{H^1_\varepsilon} \lesssim \varepsilon^{2/3} + \varepsilon^{(2N+1)/3} \|R\|^2_{H^1_\varepsilon} + \varepsilon^{4(N+1)/3} \|R\|^3_{H^1_\varepsilon}.$$

Here the norm in H^1_ε is defined by

$$\|u\|^2_{H^1_\varepsilon} := \int_{-\infty}^{\varepsilon^{-2/3}} \left[\frac{W_0(y)u(y)^2}{\sqrt{1-\varepsilon^{2/3} y}} + 4\sqrt{1-\varepsilon^{2/3} y}\,(u'(y))^2 \right] dy$$

and we show that H^1_ε is a Banach algebra with Sobolev's embedding

$$\|u\|_{L^\infty(J_\varepsilon)} \leq C \|u\|_{H^1_\varepsilon},$$

where C is ε-independent.

Grand finale

- The map
$$\Psi_\varepsilon : f \mapsto \phi := (T^\varepsilon)^{-1} \frac{f}{\sqrt{1 - \varepsilon^{2/3} y}}$$
is continuous from L^2_ε into H^1_ε and the norm of Ψ_ε is uniformly bounded in ε.
- By the Fixed Point Theorem, there exists a unique fixed point $R_{N,\varepsilon} \in H^1_\varepsilon$ such that
$$\|R_{N,\varepsilon} - R^0_{N,\varepsilon}\|_{H^1_\varepsilon} \lesssim \varepsilon^{2/3} + \varepsilon^{(2N+1)/3}.$$
- We prove that $\nu_\varepsilon(y) > 0$ for all $y \in J_\varepsilon$ so that it is the ground state η_ε by uniqueness of the positive solution η_ε.

Linearized operators

Associated with the stationary equation

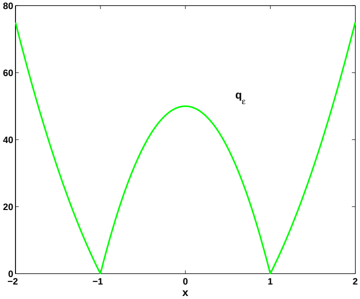
$$\varepsilon^2 \eta_\varepsilon''(x) + (1 - x^2 - \eta_\varepsilon^2(x))\eta_\varepsilon(x) = 0, \quad x \in \mathbb{R}.$$

is the linearized operator

$$L_\varepsilon = -\varepsilon^2 \partial_x^2 + V_\varepsilon(x), \quad V_\varepsilon(x) = 3\eta_\varepsilon^2(x) - 1 + x^2,$$

where

$$\lim_{\varepsilon \to 0} V_\varepsilon(x) = \begin{cases} 2(1 - x^2), & |x| \leq 1, \\ x^2 - 1, & |x| \geq 1. \end{cases}$$



Convergence of eigenvalues

Theorem

For $\varepsilon > 0$ sufficiently small, the spectrum of L_ε consists of an increasing sequence of positive eigenvalues $\{\lambda_n^\varepsilon\}_{n\geq 1}$ such that for each $n \geq 1$,

$$\lim_{\varepsilon\downarrow 0} \frac{\lambda_{2n-1}^\varepsilon}{\varepsilon^{2/3}} = \lim_{\varepsilon\downarrow 0} \frac{\lambda_{2n}^\varepsilon}{\varepsilon^{2/3}} = \mu_n,$$

where $\{\mu_n\}_{n\geq 1}$ are eigenvalues of the linearized Painlevé operator

$$M_0 u(y) := -4u''(y) + W_0(y)u(y).$$

Variational construction of excited states

The equivalent Gross–Pitaevskii equation

$$i\,\varepsilon\,\eta_\varepsilon^2 v_t + \varepsilon^2\left(\eta_\varepsilon^2 v_x\right)_x + \eta_\varepsilon^4(1-|v|^2)v = 0,$$

is the Euler–Lagrange equation for the Lagrangian $L(v) = K(v) + \Lambda(v)$ with the kinetic energy

$$K(v) = \frac{i}{2}\,\varepsilon\int_{\mathbb{R}} \eta_\varepsilon^2(x)(v\bar{v}_t - \bar{v}v_t)dx$$

and the potential energy

$$\Lambda(v) = \varepsilon^2\int_{\mathbb{R}} \eta_\varepsilon^2(x)|v_x|^2 dx + \frac{1}{2}\int_{\mathbb{R}} \eta_\varepsilon^4(x)(1-|v|^2)^2 dx.$$

If $\eta_\varepsilon \equiv 1$, the Gross–Pitaevskii equation has the exact dark soliton

$$v_1(x,t) = \sqrt{1-b^2(t)}\ \tanh\left(\varepsilon^{-1}B(t)(x-a(t))\right) + ib(t),$$

where

$$B = \frac{1}{\sqrt{2}}\sqrt{1-b^2}, \quad a = a_0 + \sqrt{2}b_0 t, \quad b = b_0.$$

Variational approximation of 1-soliton

For $\eta_\varepsilon \neq 1$, we substitute the dark soliton solution and compute the averaged Lagrangian

$$\begin{aligned} L(v_1) = & \frac{\varepsilon \dot{b}}{\sqrt{1-b^2}} \int_{\mathbb{R}} \eta_\varepsilon^2(x) \tanh(z) dx + b\sqrt{1-b^2} B \dot{a} \int_{\mathbb{R}} \eta_\varepsilon^2(x) \mathrm{sech}^2(z) dx \\ & - \varepsilon b \sqrt{1-b^2} \dot{B} B^{-1} \int_{\mathbb{R}} \eta_\varepsilon^2(x) z \,\mathrm{sech}^2(z) dx + (1-b^2) B^2 \int_{\mathbb{R}} \eta_\varepsilon^2(x) \mathrm{sech}^4(z) dx \\ & + \frac{1}{2}(1-b^2)^2 \int_{\mathbb{R}} \eta_\varepsilon^4(x) \mathrm{sech}^4(z) dx, \end{aligned}$$

where $z = \varepsilon^{-1} B(x-a)$, $B > 0$, and $a \in (-1,1)$.

Asymptotic analysis gives

$$\begin{aligned} L_1 := \lim_{\varepsilon \to 0} \frac{L(v_1)}{2\varepsilon} = & -\frac{\dot{b}}{\sqrt{1-b^2}} \left(a - \frac{1}{3} a^3\right) + b\sqrt{1-b^2}(1-a^2)\dot{a} \\ & + \frac{2}{3}(1-a^2)(1-b^2) B + \frac{1}{3B}(1-a^2)^2 (1-b^2)^2. \end{aligned}$$

Main variational result for 1-soliton

Since $\dot{B}$ is absent in $L_1 := L_1(a,b,B)$, variation of L_1 with respect to B gives

$$B = \frac{1}{\sqrt{2}}\sqrt{1-a^2}\sqrt{1-b^2}.$$

Eliminating B from $L_1(a,b,B)$, the effective Lagrangian becomes

$$L_1(a,b) = \frac{2\sqrt{2}}{3}(1-a^2)^{3/2}(1-b^2)^{3/2} - 2\sqrt{1-b^2}\dot{b}(a - \frac{1}{3}a^3).$$

The Euler–Lagrange equations are now

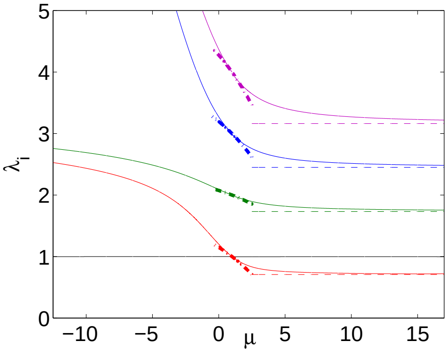
$$\dot{a} = \sqrt{2}\sqrt{1-a^2}b, \quad \dot{b} = -\frac{\sqrt{2}a(1-b^2)}{\sqrt{1-a^2}},$$

which is equivalent to the linear oscillator equation

$$\ddot{a} + 2a = 0.$$

Eigenfrequencies of 1-soliton

Recall the transformation $\mu = \frac{1}{2\varepsilon}$ and $\mathrm{Im}(\lambda) = \frac{\omega}{2}$.



P. & Kevrekidis, Cont.Math. (2008)

Lyapunov–Schmidt decomposition

The first excited state is an odd stationary solution such that

$$u_\varepsilon(0) = 0, \quad u_\varepsilon(x) > 0 \ \text{for all} \ x > 0, \quad \text{and} \quad \lim_{x\to\infty} u_\varepsilon(x) = 0.$$

Theorem

For sufficiently small $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ with properties above and there is $C > 0$ such that

$$\left\| u_\varepsilon - \eta_\varepsilon \tanh\left(\frac{\cdot}{\sqrt{2}\,\varepsilon}\right) \right\|_{L^\infty} \leq C\,\varepsilon^{2/3}.$$

In particular, the solution converges pointwise as $\varepsilon \to 0$ to

$$u_0(x) := \lim_{\varepsilon\to 0} u_\varepsilon(x) = \eta_0(x)\mathrm{sign}(x), \quad x \in \mathbb{R}.$$

Steps of the proof

Step 1: Decomposition.
We substitute

$$u_\varepsilon(x) = \eta_\varepsilon(x)\tanh\left(\frac{x}{\sqrt{2}\,\varepsilon}\right) + w_\varepsilon(x)$$

and obtain

$$L_\varepsilon w_\varepsilon = H_\varepsilon + N_\varepsilon(w_\varepsilon),$$

where

$$L_\varepsilon := -\varepsilon^2\partial_x^2 + x^2 - 1 + 3\eta_\varepsilon^2(x)\tanh^2\left(\frac{x}{\sqrt{2}\,\varepsilon}\right),$$

$$H_\varepsilon(x) := \eta_\varepsilon(x)\left(\eta_\varepsilon^2(x) - 1\right)\mathrm{sech}^2\left(\frac{x}{\sqrt{2}\,\varepsilon}\right)\tanh\left(\frac{x}{\sqrt{2}\,\varepsilon}\right) + \sqrt{2}\,\varepsilon\,\eta_\varepsilon'(x)\mathrm{sech}^2\left(\frac{x}{\sqrt{2}\,\varepsilon}\right)$$

and

$$N_\varepsilon(w_\varepsilon)(x) = -3\eta_\varepsilon(x)\tanh\left(\frac{x}{\sqrt{2}\,\varepsilon}\right) w_\varepsilon^2(x) - w_\varepsilon^3(x).$$

Steps of the proof

Step 2: Linear estimates.
Using variable $x = \sqrt{2}\,\varepsilon\, z$, we obtain

$$\hat{L}_\varepsilon = -\frac{1}{2}\partial_z^2 + 2\,\varepsilon^2\, z^2 - 1 + 3\hat{\eta}_\varepsilon^2(z)\tanh^2(z) = \hat{L}_0 + \hat{U}_\varepsilon(z),$$

where

$$\hat{L}_0 := -\frac{1}{2}\partial_z^2 + 2 - 3\mathrm{sech}^2(z)$$

and

$$\hat{U}_\varepsilon(z) := 2\,\varepsilon^2\, z^2 + 3(\hat{\eta}_\varepsilon^2(z) - 1)\tanh^2(z).$$

The spectrum of $\hat{L}_0$ consists of two eigenvalues at 0 and $\frac{3}{2}$ with eigenfunctions $\mathrm{sech}^2(z)$ and $\tanh(z)\mathrm{sech}(z)$ and the continuous spectrum on $[2,\infty)$.

Steps of the proof

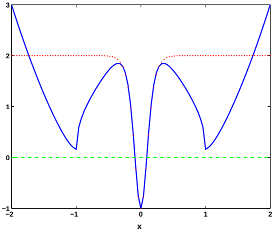


Figure: Potentials of operators L_ε (solid line) and L_0 (dots) for the first excited state.

Resolvent of the unperturbed operator:

$$\exists C > 0, \; \alpha > 0 : \quad \forall \hat{f} \in L^2_{\rm odd}(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R}) : \quad \|\hat{L}_0^{-1} \hat{f}\|_{H^2 \cap L^\infty_\alpha} \leq C \|\hat{f}\|_{L^2 \cap L^\infty_\alpha}.$$

Resolvent of the full operator:

$$\exists C > 0 : \quad \forall \hat{f} \in L^2_{\rm odd}(\mathbb{R}) : \quad \|\hat{L}_\varepsilon^{-1} \hat{f}\|_{H^2} \leq C \, \varepsilon^{-2/3} \, \|\hat{f}\|_{L^2}.$$

Steps of the proof

Step 3: Bounds on the inhomogeneous and nonlinear terms.
Recall that we are solving

$$L_\varepsilon w_\varepsilon = H_\varepsilon + N_\varepsilon(w_\varepsilon),$$

where

$$\hat{H}_\varepsilon \in L^2_{\rm odd}(\mathbb{R}) \quad \text{and} \quad \hat{N}_\varepsilon(\hat{w}_\varepsilon) : H^2_{\rm odd}(\mathbb{R}) \mapsto L^2_{\rm odd}(\mathbb{R}).$$

For any $\varepsilon > 0$ and $\alpha \in (0,2)$, we have

$$\begin{aligned} \|\hat{H}_\varepsilon\|_{L^2 \cap L^\infty_\alpha} &\leq \|\eta_\varepsilon\|_{L^\infty} \|(1-\hat{\eta}_\varepsilon^2){\rm sech}^2(\cdot)\|_{L^2\cap L^\infty_\alpha} + \sqrt{2}\,\varepsilon\,\|\eta_\varepsilon'\|_{L^\infty}\|{\rm sech}^2(\cdot)\|_{L^2\cap L^\infty_\alpha} \\ &\leq C\,\varepsilon^{2/3}. \end{aligned}$$

For any $\hat{w}_\varepsilon \in H^2(\mathbb{R})$, we have

$$\|\hat{N}_\varepsilon(\hat{w}_\varepsilon)\|_{L^2} \leq 3\|\eta_\varepsilon\|_{L^\infty}\|\hat{w}_\varepsilon^2\|_{H^2} + \|\hat{w}_\varepsilon^3\|_{H^2} \leq 3\|\hat{w}_\varepsilon\|_{H^2}^2 + \|\hat{w}_\varepsilon\|_{H^2}^3.$$

Steps of the proof

Step 4: Normal-form transformation.
Let

$$\hat{w}_\varepsilon = \hat{w}_1 + \hat{w}_2 + \hat{\varphi}_\varepsilon, \quad \hat{w}_1 = \hat{L}_0^{-1}\hat{H}_\varepsilon, \quad \hat{w}_2 = -3\hat{L}_0^{-1}\hat{\eta}_\varepsilon \tanh(z)\hat{w}_1^2,$$

where

$$\exists C > 0 : \quad \|\hat{w}_1\|_{H^2 \cap L^\infty_\alpha} \leq C\,\varepsilon^{2/3}, \quad \|\hat{w}_2\|_{H^2 \cap L^\infty_\alpha} \leq C\,\varepsilon^{4/3}.$$

The remainder term $\hat{\varphi}_\varepsilon$ solves the new problem

$$\mathcal{L}_\varepsilon \hat{\varphi}_\varepsilon = \mathcal{H}_\varepsilon + \mathcal{N}_\varepsilon(\hat{\varphi}_\varepsilon),$$

where

$$\|\mathcal{H}_\varepsilon\|_{L^2} \leq C\,\varepsilon^2,$$

$$\forall \hat{\varphi}_\varepsilon \in B_\delta(H^2_{\rm odd}) : \quad \|\mathcal{N}_\varepsilon(\hat{\varphi}_\varepsilon)\|_{L^2} \leq C(\delta)\|\hat{\varphi}_\varepsilon\|^2_{H^2},$$

and

$$\forall \hat{\varphi}_\varepsilon, \hat{\phi}_\varepsilon \in B_\delta(H^2_{\rm odd}) : \quad \|\mathcal{N}_\varepsilon(\hat{\varphi}_\varepsilon) - \mathcal{N}_\varepsilon(\hat{\phi}_\varepsilon)\|_{L^2} \leq C(\delta)\left(\|\hat{\varphi}_\varepsilon\|_{H^2} + \|\hat{\phi}_\varepsilon\|_{H^2}\right)\|\hat{\varphi}_\varepsilon - \hat{\phi}\|_{H^2}.$$

Steps of the proof

Step 5: Fixed-point arguments.
Since

$$\exists C > 0 : \quad \forall \hat{f} \in L^2_{\rm odd}(\mathbb{R}) : \quad \|\mathcal{L}_\varepsilon^{-1} \hat{f}\|_{H^2} \leq C\, \varepsilon^{-2/3} \|\hat{f}\|_{L^2},$$

the map $\hat{\varphi}_\varepsilon \mapsto \mathcal{L}_\varepsilon^{-1} \mathcal{N}_\varepsilon(\hat{\varphi}_\varepsilon)$ is a contraction in the ball $B_\delta(H^2_{\rm odd})$ if $\delta \ll \varepsilon^{2/3}$.

On the other hand, the source term $\mathcal{L}_\varepsilon^{-1} \mathcal{H}_\varepsilon$ is as small as $\mathcal{O}(\varepsilon^{4/3})$. Therefore, Banach's Fixed-Point Theorem applies in the ball $B_\delta(H^2_{\rm odd})$ with $\delta \sim \varepsilon^{4/3}$.

Step 6: Properties of $u_\varepsilon(x)$. It remains to prove that $u_\varepsilon(x) > 0$ for all $x > 0$. This property does not come immediately from the fixed-point solution

$$u_\varepsilon(x) = \eta_\varepsilon(x) \tanh\left(\frac{x}{\sqrt{2}\,\varepsilon}\right) + w_\varepsilon(x),$$

where $\|w_\varepsilon\|_{L^\infty} \leq C\, \varepsilon^{2/3}$.

Variational approximation of 2-solitons

A superposition of two dark solitons

$$\begin{aligned} v_2(x,t) &= \left[A_1(t)\ \tanh\left(\varepsilon^{-1}B_1(t)(x-a_1(t))\right)+ib_1(t)\right] \\ &\quad \times \left[A_2(t)\ \tanh\left(\varepsilon^{-1}B_2(t)(x-a_2(t))\right)+ib_2(t)\right], \qquad (1) \end{aligned}$$

where $a_j \in (-1,1)$, $b_j \in (-1,1)$, and

$$A_j = \sqrt{1-b_j^2}, \quad B_j = \frac{1}{\sqrt{2}}\sqrt{1-a_j^2}\sqrt{1-b_j^2}, \quad j = 1,2.$$

Out-of-phase oscillations for

$$a_1 = -a, \quad a_2 = a, \quad b_1 = -b, \quad b_2 = b,$$

where

$$a \leq C_1\,\varepsilon^{1/6}, \quad e^{-4Ba\,\varepsilon^{-1}} \leq C_2\,\varepsilon^2\,|\log(\varepsilon)|,$$

The first condition ensures that the dark solitons are close to the center of the harmonic potential. The second condition ensures that the overlapping between the dark solitons is small.

Averaged Lagrangian for 2-solitons

Potential energy

$$\Lambda_2 := \frac{\Lambda(v_2)}{2\,\varepsilon} = \Lambda_+ + \Lambda_- + \Lambda_{\rm overlap},$$

where

$$\lim_{\varepsilon \to 0} (\Lambda_+ + \Lambda_-) = \frac{2\sqrt{2}}{3}(1-a^2)^{3/2}(1-b^2)^{3/2}.$$

and

$$\Lambda_{\rm overlap} = -8\sqrt{2}(1-a^2)^{3/2}(1-b^2)^{5/2}\, e^{-4Ba\,\varepsilon^{-1}} \left(1 + \mathcal{O}(\varepsilon^{1/3})\right).$$

Kinetic energy

$$K_2 := \frac{K(v_2)}{2\,\varepsilon} = K_+ + K_- + K_{\rm overlap},$$

where

$$\lim_{\varepsilon \to 0} (K_+ + K_-) = -4\sqrt{1-b^2}\dot{b}(a - \frac{1}{3}a^3).$$

Main variational results for 2-solitons

In variables (a, b), the Euler–Lagrange equations at the leading order give

$$\dot{a} = \sqrt{2}b, \quad \dot{b} = -\sqrt{2}a + 8\,\varepsilon^{-1}\,e^{-2\sqrt{2}a\,\varepsilon^{-1}},$$

or, equivalently,

$$\ddot{a} + 2a = 8\sqrt{2}\,\varepsilon^{-1}\,e^{-\frac{2\sqrt{2}a}{\varepsilon}}.$$

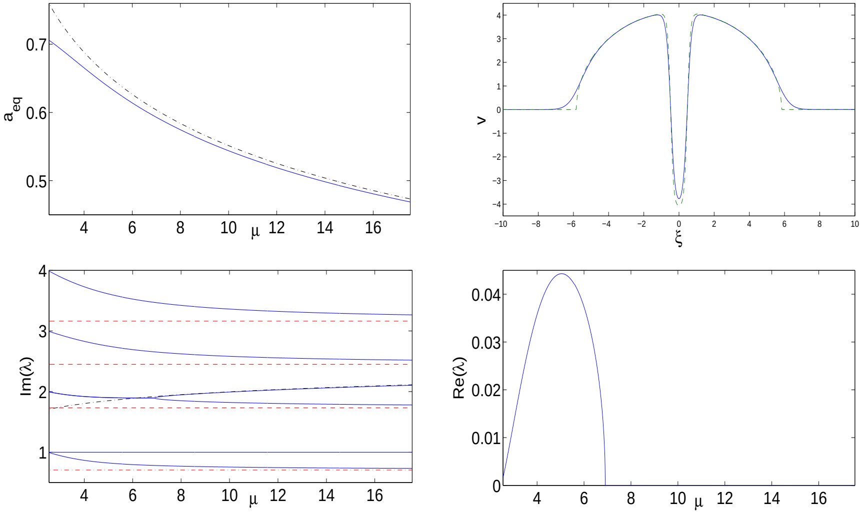
The equilibrium state $a_0(\varepsilon)$ is given asymptotically by

$$a = \frac{\varepsilon}{\sqrt{2}}\left(-\log(\varepsilon) - \frac{1}{2}\log|\log(\varepsilon)| + \frac{3}{2}\log(2) + o(1)\right) \quad \text{as} \quad \varepsilon \to 0.$$

The linear out-of-phase oscillations near the stationary state have squared frequency

$$\omega_0^2(\varepsilon) = -4\log(\varepsilon) - 2\log|\log(\varepsilon)| + 2 + 6\log(2) + o(1), \quad \text{as} \quad \varepsilon \to 0.$$

Eigenfrequencies of 2-solitons



Rigorous results

The second excited state is an odd stationary solution such that

$$u_\varepsilon(x) > 0 \ \text{ for all } \ |x| > x_0, \quad u_\varepsilon(x) < 0 \ \text{ for all } \ |x| < x_0, \quad \text{and} \quad \lim_{x\to\infty} u_\varepsilon(x) = 0.$$

Theorem

For sufficiently small $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ with properties above and there exist $a > 0$ and $C > 0$ such that

$$\left\| u_\varepsilon - \eta_\varepsilon \tanh\left(\frac{\cdot - a}{\sqrt{2}\,\varepsilon}\right) \tanh\left(\frac{\cdot + a}{\sqrt{2}\,\varepsilon}\right) \right\|_{L^\infty} \leq C\,\varepsilon^{2/3}$$

and

$$a = -\frac{\varepsilon}{\sqrt{2}} \left(\log(\varepsilon) + \frac{1}{2} \log|\log(\varepsilon)| - \frac{3}{2} \log(2) + o(1) \right) \quad \text{as} \quad \varepsilon \to 0.$$

In particular, $x_0 = a + \mathcal{O}(\varepsilon^{5/3})$.

Steps of the proof

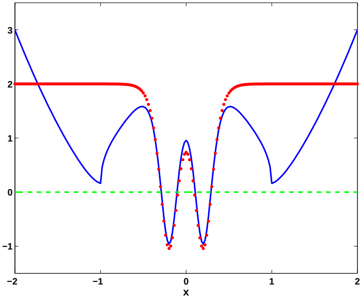


Figure: Potential of operator L_ε (solid line) and L_0 (dots) for the second excited state.

Here the leading-order operator

$$\hat{L}_0(\zeta) = -\frac{1}{2}\partial_z^2 + 2 - 3\mathrm{sech}^2(z+\zeta) - 3\mathrm{sech}^2(z-\zeta), \quad \zeta = \frac{a}{\sqrt{2}\,\varepsilon},$$

has two eigenvalues in the neighborhood of 0 for large ζ because of the double-well potential centered at $z = \pm\zeta$.

Main variational results for *m*-solitons

We can set up the leading-order averaged Lagrangian for m dark solitons:

$$L_m \sim -\sqrt{2}\sum_{j=1}^{m}(a_j^2+b_j^2) - 2\sum_{j=1}^{m} a_j \dot{b}_j - 8\sqrt{2}\sum_{j=1}^{m-1} e^{-\sqrt{2}(a_{j+1}-a_j)\,\varepsilon^{-1}},$$

which generate the Euler–Lagrangian equations

$$\ddot{a}_j + 2a_j + 8\sqrt{2}\,\varepsilon^{-1}\left(e^{-\sqrt{2}(a_{j+1}-a_j)\,\varepsilon^{-1}} - e^{-\sqrt{2}(a_j-a_{j-1})\,\varepsilon^{-1}}\right) = 0.$$

The center of mass $\langle a\rangle = \frac{1}{m}\sum_{j=1}^{m} a_j$ satisfies

$$\langle \ddot{a}\rangle + 2\langle a\rangle = 0,$$

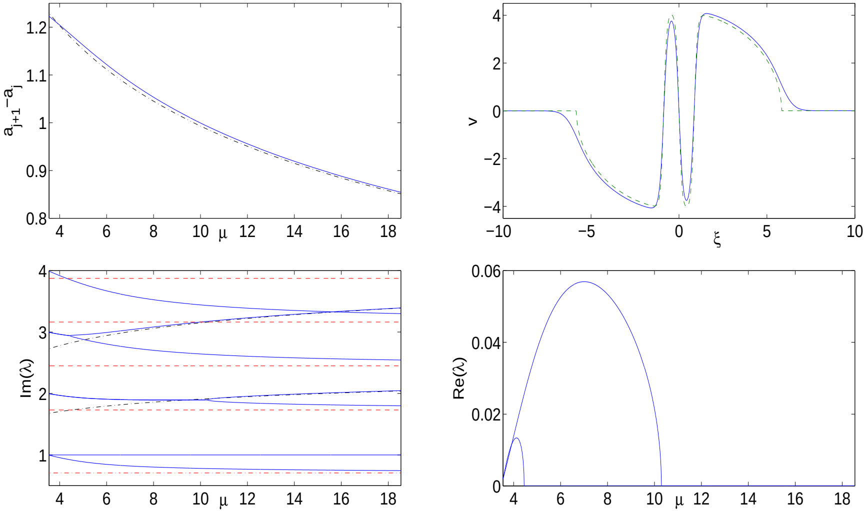
The normal coordinates

$$x_j = \sqrt{2}(a_{j+1}-a_j)\,\varepsilon^{-1}, \quad j \in \{1,2,...,m-1\},$$

satisfy

$$\ddot{x}_j + 2x_j + 16\,\varepsilon^{-2}\left(e^{-x_{j+1}} - 2e^{-x_j} + e^{-x_{j-1}}\right) = 0, \quad j \in \{1,2,...,m-1\}.$$

Eigenfrequencies of 3-solitons



Summary of our results

- We justified asymptotic representations of the ground and excited states
- We predicted asymptotic dependence of the distance between dark solitons for m-excited states.
- We predicted asymptotic dependence of the eigenfrequencies of oscillations for m-excited states related to the dynamics of dark solitons with respect to each other and to the harmonic potential.
- We illustrated both asymptotic predictions numerically.
- Analysis of vortices, dipoles, and other vortex configurations in the space of two dimensions is currently in progress.