Trapped vortices in the harmonic potentials

Dmitry Pelinovsky

Department of Mathematics, McMaster University, Hamilton, Ontario, Canada
http://dmpeli.math.mcmaster.ca

SIAM conference on analysis of PDEs, San Diego, November 16, 2011

References:

C. Gallo & D.P., Asymptotic Analysis **73**, 53-96 (2011)
M. Coles, D.P., P. Kevrekidis, Nonlinearity **23**, 1753–1770 (2010)
D.P., P. Kevrekidis, Nonlinearity **24**, 1271–1289 (2011)
D.P., Nonlinear Analysis **73**, 2631–2643 (2010)

Introduction

Density waves in cigar–shaped Bose–Einstein condensates with repulsive inter-atomic interactions and a harmonic potential are modeled by the Gross-Pitaevskii equation

$$i v_\tau = -\frac{1}{2}\nabla_\xi^2 v + \frac{1}{2}|\xi|^2 v + |v|^2 v - \mu v,$$

where μ is the chemical potential, $\xi \in \mathbb{R}^d$, and ∇_ξ^2 is the Laplacian in ξ.

Using the scaling transformation,

$$v(\xi,t) = \mu^{1/2} u(x,t), \quad \xi = (2\mu)^{1/2} x, \quad \tau = 2t,$$

the Gross–Pitaevskii equation is transformed to the semi-classical form

$$i\,\varepsilon\, u_t + \varepsilon^2\, \nabla_x^2 u + (1 - |x|^2 - |u|^2) u = 0,$$

where $\varepsilon = (2\mu)^{-1}$ is a small parameter.

Ground state

Limit $\mu \to \infty$ or $\varepsilon \to 0$ is referred to as the semi-classical or Thomas–Fermi limit. Physically, it is the limit of large density of the atomic cloud.

The ground state η_ε is the real positive solution of the stationary equation,

$$\varepsilon^2 \nabla_x^2 \eta_\varepsilon + (1 - |x|^2 - \eta_\varepsilon^2)\eta_\varepsilon = 0, \quad x \in \mathbb{R}^2.$$

Theorem (Ignat & Milot, JFA (2006))

For sufficiently small $\varepsilon > 0$, there exists a global minimizer of the Gross–Pitaevskii energy

$$E_\varepsilon(u) = \int_{\mathbb{R}^2} \left(\varepsilon^2 |\nabla_x u|^2 + (|x|^2 - 1)|u|^2 + \frac{1}{2}|u|^4 \right) dx$$

in the energy space

$$\mathcal{H}_1 = \left\{ u \in H^1(\mathbb{R}^2) : \; |x|u \in L^2(\mathbb{R}^2) \right\}.$$

Ground state in the asymptotic theory

For small $\varepsilon > 0$, the ground state $\eta_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ decays to zero as $|x| \to \infty$ faster than any exponential function

$$0 < \eta_\varepsilon(x) \leq C\,\varepsilon^{1/3} \exp\left(\frac{1-|x|^2}{4\,\varepsilon^{2/3}}\right), \quad \text{for all } \; |x| \geq 1.$$

The Thomas–Fermi approximation is

$$\eta_0(x) := \lim_{\varepsilon \to 0} \eta_\varepsilon(x) = \begin{cases} (1-|x|^2)^{1/2}, & \text{for } \; |x| < 1, \\ 0, & \text{for } \; |x| > 1, \end{cases}$$

Theorem (Gallo & D.P., AA (2011))

For sufficiently small $\varepsilon > 0$, there is $C > 0$ such that

$$\|\eta_\varepsilon - \eta_0\|_{L^\infty} \leq C\,\varepsilon^{1/3}, \quad \|\nabla_x \eta_\varepsilon\|_{L^\infty} \leq C\,\varepsilon^{-1/3}.$$

Vortices

The vortex u_ε is a complex-valued solution of the stationary equation,

$$\varepsilon^2 \nabla_x^2 u_\varepsilon + (1 - |x|^2 - |u_\varepsilon|^2) u_\varepsilon = 0, \quad x \in \mathbb{R}^2.$$

The product representation

$$u(x,t) = \eta_\varepsilon(x) v(x,t)$$

brings the Gross–Pitaevskii equation to the equivalent form

$$i \varepsilon \eta_\varepsilon^2 v_t + \varepsilon^2 \nabla_x \left(\eta_\varepsilon^2 \nabla_x v\right) + \eta_\varepsilon^4 (1 - |v|^2) v = 0,$$

where $\lim_{|x| \to \infty} |v(x)| = 1$.

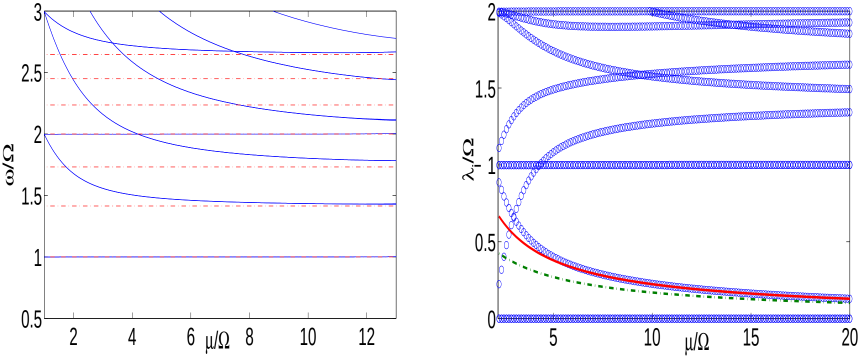
Symmetric vortex of charge $m \in \mathbb{N}$ corresponds to the choice $v = \psi(r/\varepsilon) e^{im\theta}$, where (r,θ) are polar coordinates on $\mathbb{R}^2$ and $\psi(r/\varepsilon) \to 1$ as $r \to \infty$.

Vortices in harmonic potentials

Earlier results in physics literature:

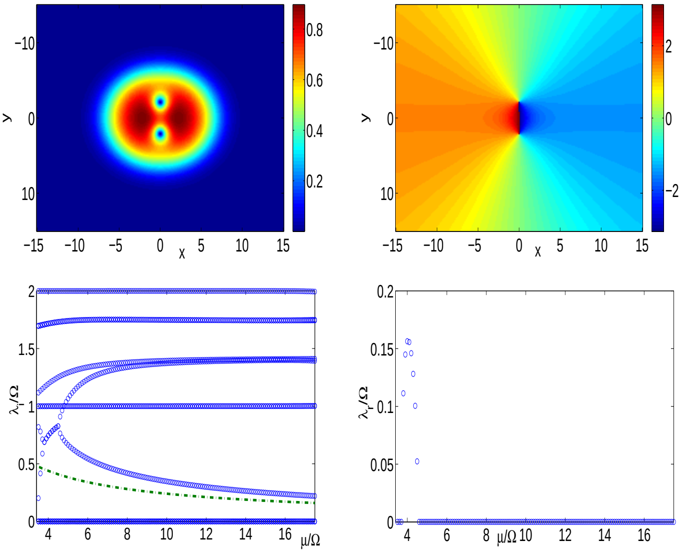
- Castin & Dum (1999) and Aftalion & Du (2001) - rotating vortices can become local and later global minimizers of energy for larger frequencies
- Fetter & Svidzinksy (2001) - vortex configurations can be understood through effective energy
- Ovchinnikov & Sigal (2004) - vortex interaction is determined by the logarithmic Kirchhoff–Onsager energy
- Möttönen *et al.* (2005) - computations of the interaction energy for two and four vortices; prediction of stationary dipoles and quadrupoles
- Li *et al.* (2008) - dynamics of a vortex–antivortex pair on a phase plane
- Middelkamp *et al.* (2010) - numerical computations of eigenvalues for single vortices, dipoles and quadrupoles by using relaxation methods
- Kollar & Pego (2011) - numerical computations of eigenvalues for charge-one and charge-two vortices by using Evans functions

1. Spectral stability of charge-one vortices

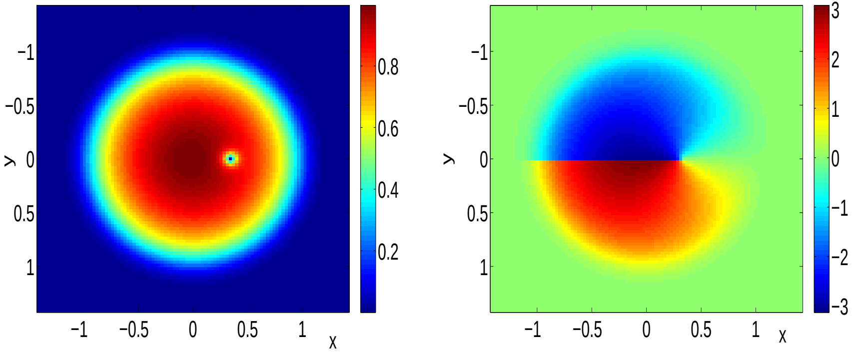


Left: ground state η_ε. Right: charge-one vortex.

2. Existence of dipole configurations



3. Steady precession of charge-one vortices



Spatial contour plots of the amplitude (left) and phase (right) of a rotating charge-one vortex.

Variational construction of vortices

The equivalent Gross–Pitaevskii equation

$$i\,\varepsilon\,\eta_\varepsilon^2 v_t + \varepsilon^2\,\nabla_x\left(\eta_\varepsilon^2 \nabla_x v\right) + \eta_\varepsilon^4 (1 - |v|^2) v = 0,$$

is the Euler–Lagrange equation for the Lagrangian $L(v) = K(v) + \Lambda(v)$ with the kinetic energy

$$K(v) = \frac{i}{2}\,\varepsilon \int_{\mathbb{R}^2} \eta_\varepsilon^2 (v \bar{v}_t - \bar{v} v_t) dx$$

and the potential energy

$$\Lambda(v) = \varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2 |\nabla_x v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \eta_\varepsilon^4 (1 - |v|^2)^2 dx.$$

Substituting a vortex ansatz for v and computing Euler–Lagrange equations for parameters of the ansatz yield the system of equations that captures qualitative dynamics of vortices and dipoles.

Free vortex of the defocusing NLS equation

As the variational ansatz, we substitute a single vortex of charge m from the defocusing NLS equation ($\eta_\varepsilon \equiv 1$),

$$V_m(x) = \Psi_m(R)e^{im\theta}, \quad R = \frac{r}{\varepsilon}$$

where $m \in \mathbb{N}$ and $\Psi_m(R)$ is a solution of

$$\Psi_m'' + R^{-1}\Psi_m' - m^2R^{-2}\Psi_m + (1 - \Psi_m^2)\Psi_m = 0, \quad R > 0,$$

such that $\Psi_m(0) = 0$, $\Psi_m(R) > 0$ for all $R > 0$, and $\lim_{R\to\infty} \Psi_m(R) = 1$.

The short-range asymptotics is

$$\Psi_m(R) = \alpha_m R^m + \mathcal{O}(R^{m+2}) \quad \text{as} \quad R \to 0$$

The long-range asymptotics is

$$\Psi_m^2(R) = 1 - \frac{m^2}{R^2} + \mathcal{O}\left(\frac{1}{R^4}\right) \quad \text{as} \quad R \to \infty.$$

Kinetic energy

We can use variables

$$x = x_0 + \varepsilon X, \quad y = y_0 + \varepsilon Y,$$

and write the kinetic energy as

$$K(V_m) = -\dot{x}_0 K_x(V_m) - \dot{y}_0 K_y(V_m),$$

where

$$K_x(V_m) = -m\,\varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \frac{Y \Psi_m^2}{R^2} dXdY, \quad K_y(V_m) = m\,\varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \frac{X \Psi_m^2}{R^2} dXdY.$$

Lemma (D.P. & P.Kevrekidis, Nonlinearity (2011))

For small $\varepsilon > 0$ and small $(x_0, y_0) \in \mathbb{R}^2$, the kinetic energy of a single vortex is represented by

$$K(V_m) = \pi m\,\varepsilon(x_0\dot{y}_0 - y_0\dot{x}_0)\left(1 + \mathcal{O}(\varepsilon) + \mathcal{O}(x_0^2 + y_0^2)\right).$$

Potential energy

We write the potential energy as

$$\Lambda(V_m) = \varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \left[\left(\frac{d\Psi_m}{dR} \right)^2 + \frac{m^2}{R^2} \Psi_m^2 \right] dXdY + \frac{\varepsilon^2}{2} \int_{\mathbb{R}^2} \eta_\varepsilon^4(x)(1 - \Psi_m^2)^2 dXdY.$$

Lemma (D.P. & P.Kevrekidis, Nonlinearity (2011))

For small $\varepsilon > 0$ and small $(x_0, y_0) \in \mathbb{R}^2$, the potential energy of a single vortex is represented by

$$\Lambda(V_m) - \Lambda(V_m)|_{x_0=y_0=0} = -\pi\,\varepsilon\, m\omega_m(x_0^2 + y_0^2) \left(1 + \mathcal{O}(\varepsilon^{1/3}) + \mathcal{O}(x_0^2 + y_0^2) \right),$$

where ω_m is given by

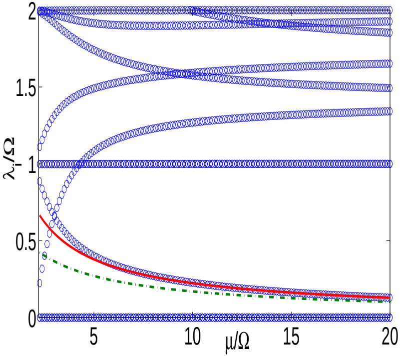
$$\omega_m = \varepsilon\, m \left[2\log(1/\varepsilon) + 1 + \frac{2}{m^2} \int_0^\infty \left[\left(\frac{d\Psi_m}{dR} \right)^2 + \frac{m^2}{R^2} \left(\Psi_m^2 - \frac{R^2}{1+R^2} \right) \right] RdR \right].$$

Eigenfrequencies of the charge-one vortex

Euler–Lagrange equations for the leading part of $L(V_m) = K(V_m) + \Lambda(V_m)$ give

$$-\dot{x}_0 = \omega_m y_0, \quad \dot{y}_0 = \omega_m x_0,$$

where $\omega_m = 2\,\varepsilon\, m|\log(\varepsilon)| + \mathcal{O}(\varepsilon)$.



Here $\mu = \frac{1}{2\,\varepsilon}$ and $\mathrm{Im}(\lambda) = \frac{\omega}{2}$.

From variational to rigorous results

A vortex of charge one has frequency $\omega_1(\varepsilon)$,

$$\omega_1(\varepsilon) = 2\,\varepsilon\,|\log(\varepsilon)| + \mathcal{O}(\varepsilon),$$

which corresponds to its periodic precession around the origin $(0,0) \in \mathbb{R}^2$ with an infinitesimal displacement from the origin.

Q: Can we find a steadily rotating vortex displaced from the origin at a small but finite distance?

Q: If we can, is this steadily rotating vortex more stable or less stable than the symmetric vortex at the origin?

Steadily rotating vortices

In the rotating coordinate frame,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad \omega \in \mathbb{R},$$

the Gross–Pitaevskii equation takes the form,

$$i\,\varepsilon\, u_t + \varepsilon^2 (u_{\xi\xi} + u_{\eta\eta}) + (1 - \xi^2 - \eta^2 - |u|^2) u + i\,\varepsilon\,\omega(\xi u_\eta - \eta u_\xi) = 0.$$

The symmetric vortex of charge one is now given by

$$u(\xi,\eta) = \sqrt{1 - \varepsilon\,\omega}\,\psi(R) e^{i\theta}, \quad \sqrt{\xi^2 + \eta^2} = \sqrt{1 - \varepsilon\,\omega} R,$$

where $\psi(R)$ satisfies,

$$\nu^2 \left(\frac{d^2\psi}{dR^2} + \frac{1}{R}\frac{d\psi}{dR} - \frac{\psi}{R^2} \right) + (1 - R^2 - \psi^2)\psi = 0, \quad \nu = \frac{\varepsilon}{1 - \varepsilon\,\omega},$$

the same equation as in the non-rotating frame.

What do we know about vortices of charge one?

- Existence $U = \psi(R)$ for any $\nu \in (0, 1/4)$ as a minimizer of

$$E_1(U) = \int_0^\infty \left[\nu^2 \left(\frac{dU}{dR} \right)^2 + \frac{\nu^2 U^2}{R^2} + R^2 U^2 + \frac{1}{2}(1 - U^2)^2 \right] R dR.$$

- $u = \psi(R) e^{i\theta}$ is a saddle point of the full energy

$$E(u) = \int_{\mathbb{R}^2} \left(\nu^2 |\nabla_x u|^2 + |x|^2 |u|^2 + \frac{1}{2}(1 - |u|^2)^2 \right) dx$$

with exactly one direction where $E(u) < E(\psi e^{i\theta})$,

$$E(\tilde{\psi} e^{i\theta}) - E(\psi e^{i\theta}) = -\pi \nu \omega_1(\nu)(x_0^2 + y_0^2) \left(1 + \mathcal{O}(\nu^{1/3} + x_0^2 + y_0^2) \right),$$

where $\omega_1(\nu) = 2\nu \log(1/\nu) + \mathcal{O}(1)$.

- The vortex is spectrally stable for any $\nu \in (0, 1/4)$.

Two linearizations

If we substitute $u(\xi,\eta,t) = \psi(r)e^{i\theta} + U(\xi,\eta,t)$ to the Gross–Pitaevskii equation with

$$U(\xi,\eta,t) = \sum_{m\in\mathbb{Z}} V^{(m)}(r)e^{im\theta}e^{\lambda t}, \quad \bar{U}(\xi,\eta,t) = \sum_{m\in\mathbb{Z}} W^{(m)}(r)e^{im\theta}e^{\lambda t},$$

then we end up with the spectral stability problem

$$H_\omega^{(m)} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix} = i\,\varepsilon\,\lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix},$$

where

$$H_\omega^{(m)} = \begin{bmatrix} 1 - r^2 + \varepsilon^2\,\Delta_m - \varepsilon\,\omega m - 2\psi^2 & -\varphi_\omega^2 \\ -\psi^2 & 1 - r^2 + \varepsilon^2\,\Delta_{m-2} + \varepsilon\,\omega(m-2) - 2\psi^2 \end{bmatrix}$$

On the other hand, linearization of the stationary problem is related to the spectrum of the self-adjoint eigenvalue problem

$$H_\omega^{(m)} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix} = \varepsilon\,\mu \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix}.$$

If $\mu = 0$ is an eigenvalue, then a bifurcation of stationary vortices occurs.

Transformation

Adopting new variables $r = \sqrt{1 - \varepsilon\,\omega} R$ and $\nu = \varepsilon/(1 - \varepsilon\,\omega)$, we transform the self-adjoint eigenvalue problem to the form,

$$H_m \left[\begin{array}{c} V_m \\ W_{m-2} \end{array} \right] = \nu\mu \left[\begin{array}{c} V_m \\ W_{m-2} \end{array} \right] + \nu\omega(m-1) \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] \left[\begin{array}{c} V_m \\ W_{m-2} \end{array} \right],$$

where

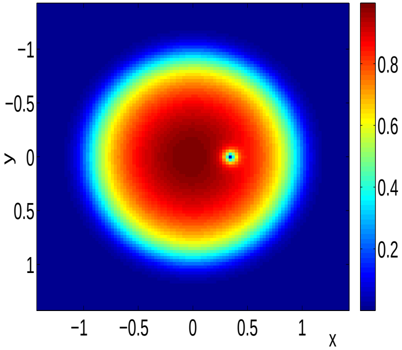
$$H_m = \left[\begin{array}{cc} 1 - R^2 + \nu^2 \Delta_m - 2\psi^2 & -\psi^2 \\ -\psi^2 & 1 - R^2 + \nu^2 \Delta_{m-2} - 2\psi^2 \end{array} \right].$$

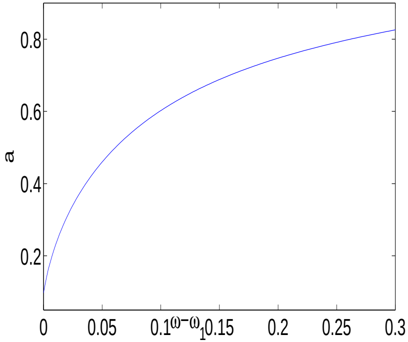
Lemma

For $m = 2$, there exists a bifurcation $\mu = 0$ at $\omega = \omega_1(\varepsilon) \approx 2\,\varepsilon\,|\log(\varepsilon)|$. Moreover, if $\mu(\omega_1(\varepsilon)) = 0$, then $\mu'(\omega_1(\varepsilon)) < 0$.

Further facts about rotating vortices

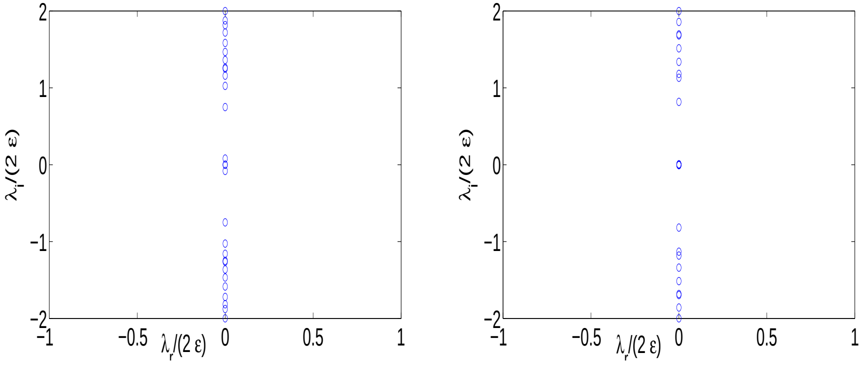
Rotating vortex is born via the supercritical pitchfork bifurcation with radial symmetry for $\omega > \omega_1$. Its center is placed at a point on the circle of radius *a* on the (ξ, η)-plane, where $a \sim \sqrt{\varepsilon(\omega - \omega_1)}$.





Further facts about rotating vortices

Symmetric vortex of charge one becomes a local minimizer of energy for $\omega > \omega_1$. Asymmetric vortex of charge one is a saddle point of energy. Nevertheless, both vortices are spectrally stable with respect to time-dependent perturbations.



Left: eigenvalues of symmetric vortex.
Right: eigenvalues of the asymmetric vortex.

Conclusion and open questions

We have discussed variational results for oscillations of vortices and dipoles in the Thomas–Fermi limit and bifurcation results for the birth of stable rotating asymmetric vortices of charge one.

Symmetric vortices of charge one are minimizers of energy and asymmetric vortices of charge one are saddle points of energy for large frequencies.

- Is the role of these vortices different in the nonlinear dynamics of the Gross–Pitaevskii equation?
- Can we explain surprising spectral stability of vortices of both types?
- Can similar bifurcations occur for dipoles and quadrupoles?