Existence and stability of standing waves for the NLS equation on a tadpole graph

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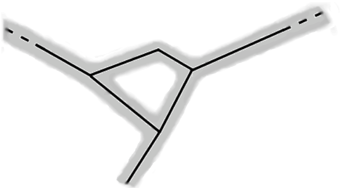
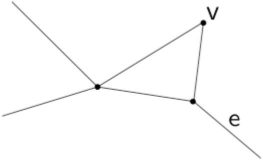
Metric Graphs

Graphs are one-dimensional approximations for constrained dynamics in which transverse dimensions are small with respect to longitudinal ones.



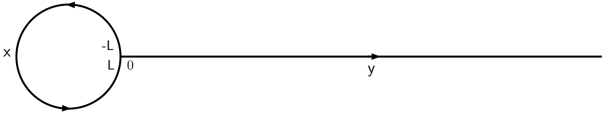


A metric graph is realized by a set of edges and vertices, with a metric structure on each edge. Proper boundary conditions are needed on the vertices to ensure that certain differential operators defined on graphs are self-adjoint.

Kirchhoff boundary conditions:

- Functions in each edge have the same value at each vertex.
- Sum of fluxes (signed derivatives of functions) is zero at each vertex.

Tadpole Graph



The ring is placed on the interval $[-L, L]$ and the semi-infinite interval is $[0, \infty)$. The Laplacian operator is defined by

$$\Delta\Psi = \left[\begin{array}{ll} u''(x), & x \in (-L, L) \\ v''(y), & y \in (0, \infty) \end{array} \right],$$

acting on functions in the form

$$\Psi = \left[\begin{array}{ll} u(x), & x \in (-L, L) \\ v(y), & y \in (0, \infty) \end{array} \right],$$

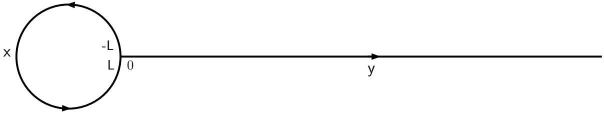
in the domain

$$\mathcal{D}(\Delta) = \left\{ \begin{array}{l} (u, v) \in H^2(-L, L) \times H^2(0, \infty) : \\ u(L) = u(-L) = v(0), \quad u'(L) - u'(-L) = v'(0) \end{array} \right\}.$$

NLS on star graphs

- Gnutzmann-Smilansky-Derevyanko, Phys. Rev. A **83** (2011), 033831: a complex set of resonances after inserting a single nonlinear edge in a linear quantum graph; recent rigorous analysis by L.Tentarelli, arXiv:1503.00455.
- Series of papers on star graphs by Adami-Cacciapuoti-Finco-Noja: Scattering of solitons; Standing waves and stability (2011-14).
- Recent work by Adami-Serra-Tilli on nonexistence of ground states in networks with closed cycles, Calc Var PDE (2015)
- Results on dispersive estimates on trees (including star graphs) in V.Banica-L.Ignat (2011-2014).
- Classification of standing waves and computations of the bifurcation diagram on tadpole graphs by C.Cacciapuoti, D.Finco, D.Noja, Phys. Rev. E **91**, 013206 (2015); rigorous results on existence, bifurcations, and stability by D.Noja, D.P., and G.Shaikhova, Nonlinearity (2015).

NLS on a tadpole graph



NLS on a tadpole graph

$$i\frac{\partial}{\partial t}\Psi = \Delta\Psi + (p+1)|\Psi|^{2p}\Psi, \quad \Psi \in \mathcal{D}(\Delta),$$

where $p > 0$ is the parameter for the power nonlinearity. The power nonlinearity is to be defined "edge by edge".

This is an example of interaction between NLS dynamics on a bounded and unbounded sets. Although it is special, it highlights interesting behavior.

Problems:

- Existence and bifurcations of standing wave solutions $\Psi = \Phi(x)e^{i\omega t}$

$$-\Delta\Phi - (p+1)|\Phi|^{2p}\Phi = \omega\Phi \qquad \omega \in \mathbb{R}\,,\ \Phi \in \mathcal{D}(\Delta).$$

- Spectral and orbital stability of standing waves.

Existence of Standing waves

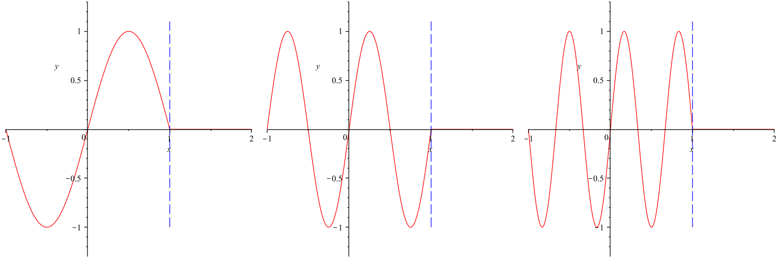
$$\begin{cases} -u''(x) - (p+1)|u|^{2p}u = \omega u, & x \in (-L, L) \ , \\ -v''(y) - (p+1)|v|^{2p}v = \omega v, & y \in (0, \infty) \ , \\ u(L) = u(-L) = v(0) \ , \\ u'(L) - u'(-L) = v'(0) \ . \end{cases}$$

Linear spectrum:

- Essential spectrum: $\sigma_{ess}(-\Delta) = [0, \infty)$ with resonance at 0.
- Embedded eigenvalues: $\left\{ \lambda_n = \left(\frac{n\pi}{L}\right)^2, \ n \in \mathbb{N} \right\} \subset \sigma_{ess}(-\Delta)$

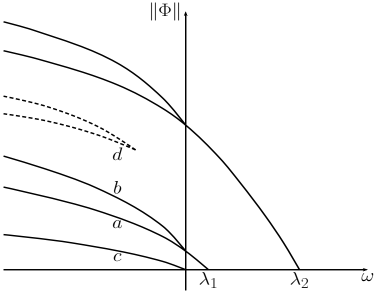
  The corresponding (normalized) eigenfunctions are:

$$\Upsilon_n = \frac{1}{\sqrt{L}} \left(\sin\left(\frac{n\pi x}{L}\right), 0 \right) \qquad n = 1, 2, 3, \ldots$$



Existence of Standing waves

The following bifurcation diagram has been computed for $p = 1$ (Cacciapuoti *et al.*, 2015):



The diagram describes the families of stationary states and their possible relation with the spectrum of $-\Delta$.

The model, although simple, exhibits a surprisingly rich behavior

- branches of standing waves bifurcating from the embedded eigenvalues
- pitchfork bifurcation at threshold $\omega = 0$: edge solitons
- saddle-node bifurcations of standing waves (dashed lines)

Standing waves bifurcating from the embedded eigenvalues

Invariant reduction

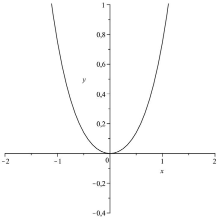
$$\begin{cases} -u''(x) - (p+1)|u|^{2p}u = \omega u, \quad x \in (-L, L), \\ u(L) = u(-L) = 0, \\ u'(L) = u'(-L). \end{cases}$$

Associated energy invariant

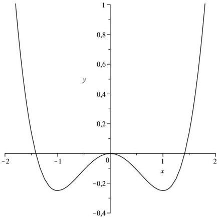
$$E = \left(\frac{du}{dx}\right)^2 + \left(\omega + |u|^{2p}\right) u^2 = \text{const.}$$

For a given L, there exist two solutions $u^{\pm}_{n,\omega}$ in $H^2_{\text{per,odd}}(-L, L)$, $n \in \mathbb{N}$ for every $\omega \in (-\infty, \lambda_n)$, where $\lambda_n := \left(\frac{n\pi}{L}\right)^2$. The map $\omega \mapsto u^{\pm}_{n,\omega}$ is C^1 in ω.

Depending on the sign of ω the effective potential V has two different forms.



$\omega > 0$



$\omega < 0$

Numerical solutions for $p = 1$

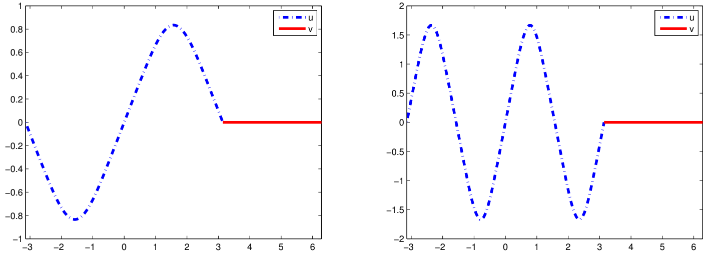


Figure : Standing wave solutions $(u^+_{n,\omega}, 0)$ versus x for $n = 1$ (a) and $n = 2$ (b) corresponding to $\omega = -1$.

Standing waves bifurcating from the zero resonance

Let $\omega = -\epsilon^2$ and consider small values of ϵ. For the solution on the tail of the tadpole, we can scale

$$v(x) = \epsilon^{\frac{1}{p}} \phi(z), \quad z = \epsilon y,$$

where ϕ is a decaying solution of the second-order equation

$$-\phi''(z) + \phi - (p+1)|\phi|^{2p}\phi = 0, \quad z > 0.$$

Let $\phi_0(z) = \mathrm{sech}^{\frac{1}{p}}(pz)$ be the unique symmetric solitary wave. Then, $\phi(z) = \phi_0(z+a)$ for unknown parameter a.

Bifurcation problem:

$$\left\{ \begin{array}{l} -u''(x) + \epsilon^2 u - (p+1)|u|^{2p}u = 0, \quad x \in (-L, L), \\ u(L) = u(-L) = \epsilon^{\frac{1}{p}} \phi_0(a), \\ u'(L) - u'(-L) = \epsilon^{1+\frac{1}{p}} \phi_0'(a). \end{array} \right.$$

- Primary branch (positive definite) bifurcating from zero solution.
- Higher branches (sign-indefinite) bifurcating from solutions $u^{\pm}_{n,\omega}$.

Primary branch

Using the scaling transformation

$$u(x) = \epsilon^{\frac{1}{p}} \psi(z), \quad z = \epsilon x,$$

we can write the bifurcation problem as

$$\begin{cases} -\psi''(z) + \psi - (p+1)|\psi|^{2p}\psi = 0, \quad z \in (-\epsilon L, \epsilon L), \\ \psi(\epsilon L) = \psi(-\epsilon L) = \phi_0(a), \\ \psi'(\epsilon L) - \psi'(-\epsilon L) = \phi_0'(a). \end{cases}$$

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Construction of positive solution:

- An even solution is defined near the origin: $\psi(0) = \psi_0$, $\psi'(0) = 0$.
- The continuity boundary condition

  $$\phi_0(a) = \psi(\epsilon L) = \psi_0 + \frac{1}{2}\psi''(0)\epsilon^2 L^2 + \mathcal{O}(\epsilon^4),$$

  hence $\psi_0 = \phi_0(a) + \mathcal{O}(\epsilon^2)$ is uniquely defined for every $a \in \mathbb{R}$.
- The flux boundary condition

  $$\phi_0'(a) = 2\psi'(\epsilon L) = 2\psi''(0)\epsilon L + \mathcal{O}(\epsilon^3).$$

  where $\phi_0'(0) = 0$ and $\phi_0''(0) \neq 0$. Hence, $a = 2\epsilon L + \mathcal{O}(\epsilon^3)$ is unique.
- The small solution is unique and positive: $\psi = 1 + \mathcal{O}_{C^\infty(-L,L)}(\epsilon^2)$.

Higher branches

Bifurcation problem:

$$\begin{cases} -u''(x) + \epsilon^2 u - (p+1)|u|^{2p} u = 0, \quad x \in (-L, L), \\ u(L) = u(-L) = \epsilon^{\frac{1}{p}} \phi_0(a), \\ u'(L) - u'(-L) = \epsilon^{1+\frac{1}{p}} \phi_0'(a). \end{cases} \tag{1}$$

Since $u \approx u^{\pm}_{n,\omega=-\epsilon^2} \neq 0$, no scaling transformation can be used.

Higher branches

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$$\left\{ \begin{array}{l} -u''(x) + \epsilon^2 u - (p+1)|u|^{2p} u = 0, \quad x \in (-L, L), \\ u(L) = u(-L) = \epsilon^{\frac{1}{p}} \phi_0(a), \\ u'(L) - u'(-L) = \epsilon^{1+\frac{1}{p}} \phi_0'(a). \end{array} \right. \tag{1}$$

Since $u \approx u^{\pm}_{n,\omega=-\epsilon^2} \neq 0$, no scaling transformation can be used.

Construction of particular solutions:

- Take $u_\epsilon := u^{\pm}_{n,\omega=-\epsilon^2}$ and translate as $u(x) = u_\epsilon(x+b)$ for $b \in \mathbb{R}$.
- Since u_ϵ is $2L$-periodic, the flux boundary condition is satisfied if $a = 0$.
- The continuity boundary conditions yields

$$u_\epsilon(L+b) = \epsilon^{\frac{1}{p}}, \quad \phi_0(0) = 1.$$

  Since $u_\epsilon(L) = 0$ and $u_\epsilon'(L) \neq 0$, $b = \frac{1}{u_\epsilon'(L)} \epsilon^{\frac{1}{p}} + \mathcal{O}\left(\epsilon^{\frac{3}{p}}\right)$ is unique.
- Pitchfork bifurcation: two different solutions exist for $u_\epsilon'(L) \gtrless 0$.
- Uniqueness of solution $u \approx u_\epsilon$ can be shown.

Numerical solutions for $p = 1$

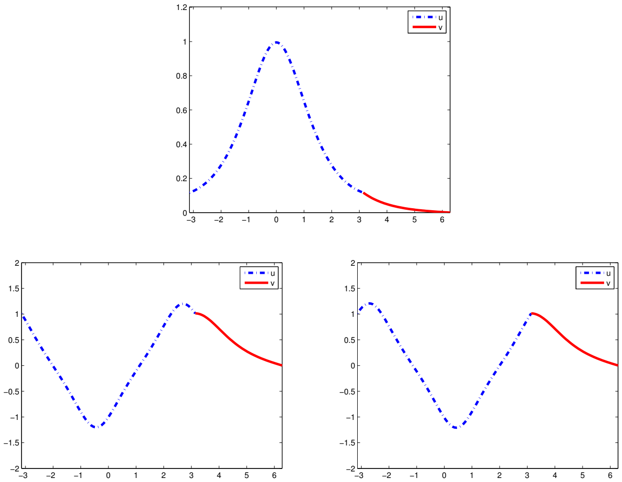
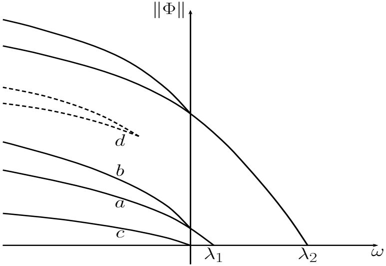


Figure : Standing wave solutions (u, v) versus x for $\omega = -1$ along the primary branch (a) and the higher branches for $n = 1$ (b,c).

Summary on existence of standing waves

1. A continuous branch bifurcating from the zero energy resonance of the system
2. A continuous branch of edge solitons displaying a pitchfork bifurcation at the threshold of the continuous spectrum



Open problems:

- Continuation of new branches in ω to $\omega \to -\infty$
- Prediction of saddle-node bifurcations for dashed branches.

Orbital stability

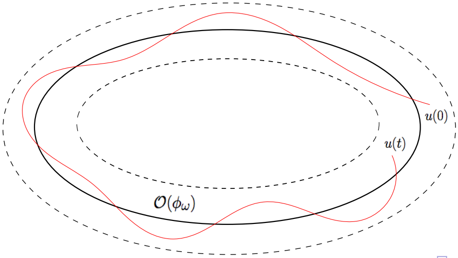
Recall that NLS has $U(1)$, or phase, symmetry. No stability of equilibrium *points* Φ can hold, but stability of equilibrium *orbits* $\{e^{i\theta}\Phi\}_{\theta\in\mathbb{R}}$ may be attained in some cases.

Definition

We say that $e^{i\omega t}\Phi$ is *orbitally stable* in a Banach space V if $\forall\epsilon > 0\ \exists\delta > 0$ such that $\forall u_0 \in V$ with $\|u_0 - \Phi\|_V < \delta$, NLS has a global solution $u(t) \in V$ with initial datum u_0 satisfying

$$\inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\Phi\|_V < \epsilon,$$

for every $t \in \mathbb{R}$.



Spectral stability

Linearization of NLS with $\Psi = e^{i\omega t}\left(\Phi(x) + U(x,t) + iW(x,t)\right)$ and the separation of variables $U(x,t) = \tilde{U}(x)e^{\lambda t}$ results in the spectral problem

$$L_+\tilde{U} = -\lambda\tilde{W}, \quad L_-\tilde{W} = \lambda\tilde{U},$$

associated with the Schrödinger operators L_+ and L_-. For the tadpole graph, this yields two self-adjoint problems

$$L_- : \quad \begin{cases} -U''(x) - \omega U - (p+1)|u|^{2p}U = \lambda U, & x \in (-L, L), \\ -V''(y) - \omega V - (p+1)|v|^{2p}V = \lambda V, & y \in (0,\infty), \\ U(L) = U(-L) = V(0), \\ U'(L) - U'(-L) = V'(0), \end{cases}$$

and

$$L_+ : \quad \begin{cases} -U''(x) - \omega U - (2p+1)(p+1)|u|^{2p}U = \lambda U, & x \in (-L, L), \\ -V''(y) - \omega V - (2p+1)(p+1)|v|^{2p}V = \lambda V, & y \in (0,\infty), \\ U(L) = U(-L) = V(0), \\ U'(L) - U'(-L) = V'(0). \end{cases}$$

Definition

We say that the standing wave is spectrally unstable if there exist an eigenvector $\tilde{U}, \tilde{W} \in \mathcal{D}(-\Delta)$ for an eigenvalue with $\mathrm{Re}(\lambda) > 0$. Otherwise, we say that it is weakly spectrally stable.

Criteria for spectral stability

Constrained space associated with the $U(1)$, or phase, symmetry:

$$L^2_c := \left\{ U \in L^2 : \quad \langle U, \Phi \rangle_{L^2} = 0 \right\}.$$

Denote the number of negative eigenvalues of $L_\pm$ by $n(L_\pm)$ and assume that L_+ is invertible and $\mathrm{Ker}(L_-)$ is one-dimensional.

The following criteria summarize the results from Shatah–Straus (1983), Weinstein (1985), Grillakis (1990), Jones (1990), Kapitula–Kevrekidis–Stanstede (2004), P. (2005), etc.

- If $n(L_+) = 1$ and $n(L_-) = 0$, then
  - Φ is spectrally and orbitally stable if $n(L_+|_{L^2_c}) = 0$
  - Φ is unstable if $n(L_+|_{L^2_c}) = 1$.
- If $n(L_+|_{L^2_c}) - n(L_-)$ is nonzero, then Φ is unstable.
- If $n(L_+|_{L^2_c}) + n(L_-)$ is odd, then Φ is unstable.
- If $n(L_+|_{L^2_c}) + n(L_-)$ is even, then Φ is stable if there exist $n(L_+|_{L^2_c}) + n(L_-)$ eigenvalues $\lambda \in i\mathbb{R}$ of negative Krein signature.

Stability of the primary branch for $\omega = -\epsilon^2$

Recall that

$$\Phi = \epsilon^{\frac{1}{p}}(\psi(z), \phi(z)), \quad z = \epsilon x,$$

where $\psi(z) = 1 + \mathcal{O}(z^2)$ and $\phi(z) = \phi_0(z + a)$.

- $n(L_-) = 0$ because Φ is strictly positive.
- $n(L_+) = 1$ because of the reduction to the scalar Schrödinger equation.

The spectral problem for L_+ with $\lambda = \epsilon^2 \Lambda$ is

$$\left\{ \begin{array}{ll} -U''(z) + U(z) - (2p+1)(p+1)|\psi(z)|^{2p} U(z) = \Lambda U(z), & z \in (-\epsilon L, \epsilon L), \\ -V''(z) + V(z) - (2p+1)(p+1)|\phi(z)|^{2p} V(z) = \Lambda V(z), & z \in (0, \infty), \\ U(\epsilon L) = U(-\epsilon L) = V(0), & \\ U'(\epsilon L) - U'(-\epsilon L) = V'(0), & \end{array} \right.$$

The leading-order spectral problem is related to the scalar Schrödinger equation on the line

$$\left\{ \begin{array}{ll} -V''(z) + V(z) - (2p+1)(p+1)\mathrm{sech}^2(pz) V(z) = \Lambda V(z), & z \in (0, \infty), \\ V'(0) = 0, & \end{array} \right.$$

which has only one negative eigenvalue $\Lambda_0 < 0$.

Stability of the primary branch

- $n(L_+|_{L^2_c}) = 0$ if the slope condition is satisfied

$$\frac{d}{d\omega}\|\Phi\|^2 < 0$$

This can be checked directly from asymptotic solutions:

$$\|u\|^2_{L^2(-L,L)} = \epsilon^{\frac{2}{p}}\|\psi(\epsilon\cdot)\|^2_{L^2(-L,L)} = \epsilon^{\frac{2}{p}}\left(2L + \mathcal{O}(\epsilon^2)\right)$$

and

$$\|v\|^2_{L^2(0,\infty)} = \epsilon^{\frac{2}{p}-1}\|\phi_0\|^2_{L^2(a,\infty)} = \epsilon^{\frac{2}{p}-1}\left(\|\phi_0\|^2_{L^2(0,\infty)} + \mathcal{O}(\epsilon)\right).$$

Theorem

For $\omega = -\epsilon^2$ with $\epsilon > 0$ sufficiently small, the primary branch is orbitally stable for every $p \in (0,2)$ and orbitally unstable for every $p \in (2,\infty)$.

Open problems:

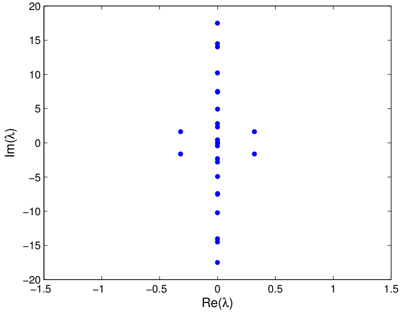
- For $p = 1$, show that Φ is a constrained minimizer of S_ω, $\omega < 0$.
- For $p = 2$, the test for orbital stability is inconclusive.

Stability of the higher branches for $\omega = -\epsilon^2$

The degenerate higher branches are:

$$\Phi = (u^{\pm}_{n,\omega}(x+b), 0).$$

Numerical solutions for $p = 1$



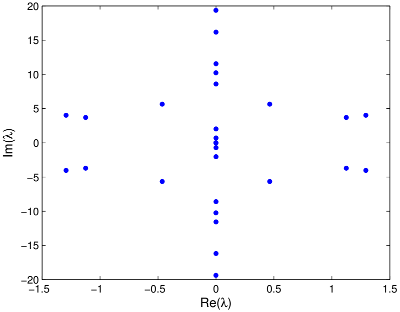


Figure : Eigenvalues λ of the spectral stability problem on the complex plane for the standing wave solutions $(u^{+}_{n,\omega}, 0)$ with $n = 1$ (a) and $n = 2$ (b) for $\omega = -1$.

Conclusions and Perspectives

- The classification of nonlinear bound states for the cubic NLS equation on a tadpole graph exhibits behavior previously unknown for the standard NLS equation with power nonlinearity on the line.

- The analysis suggests soliton bifurcations from the edge of the continuum spectrum is a general feature when stationary states on a bounded interval are coupled with stationary states on the unbounded interval. Stability properties are accessible near the bifurcation.

- Complete the stability analysis:
  - Energy minimization properties for the primary branch
  - Spectral stability along the higher branches as $\omega \to -\infty$

- Other graphs: influence of geometry (or more complex topology)?