

# Existence and stability of standing waves for the NLS equation on a tadpole graph

**Dmitry Pelinovsky**

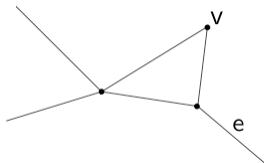
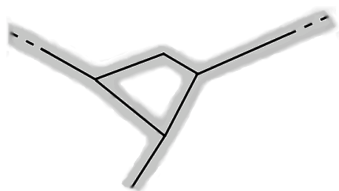
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# Metric Graphs

Graphs are one-dimensional approximations for constrained dynamics in which **transverse dimensions are small with respect to longitudinal ones**.

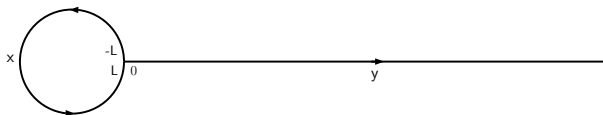


A metric graph is realized by a set of edges and vertices, with a metric structure on each edge. Proper boundary conditions are needed on the vertices to ensure that certain differential operators defined on graphs are self-adjoint.

## Kirchhoff boundary conditions:

- ▶ Functions in each edge have the same value at each vertex.
- ▶ Sum of fluxes (signed derivatives of functions) is zero at each vertex.

## Tadpole Graph



The ring is placed on the interval  $[-L, L]$  and the semi-infinite interval is  $[0, \infty)$ . The Laplacian operator is defined by

$$\Delta\Psi = \begin{bmatrix} u''(x), & x \in (-L, L) \\ v''(y), & y \in (0, \infty) \end{bmatrix},$$

acting on functions in the form

$$\Psi = \begin{bmatrix} u(x), & x \in (-L, L) \\ v(y), & y \in (0, \infty) \end{bmatrix},$$

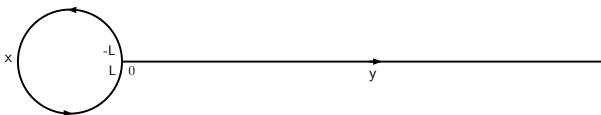
in the domain

$$\mathcal{D}(\Delta) = \left\{ (u, v) \in H^2(-L, L) \times H^2(0, \infty) : \begin{array}{l} u(L) = u(-L) = v(0), \\ u'(L) - u'(-L) = v'(0) \end{array} \right\}.$$

## NLS on star graphs

- ▶ Gnutzmann-Smilansky-Derevyanko, Phys. Rev. A **83** (2011), 033831: a complex set of resonances after inserting a single nonlinear edge in a linear quantum graph; recent rigorous analysis by L.Tentarelli, arXiv:1503.00455.
- ▶ Series of papers on star graphs by Adami-Cacciapuoti-Finco-Noja: Scattering of solitons; Standing waves and stability (2011-14).
- ▶ Recent work by Adami-Serra-Tilli on nonexistence of ground states in networks with closed cycles, Calc Var PDE (2015)
- ▶ Results on dispersive estimates on trees (including star graphs) in V.Banica-L.Ignat (2011-2014).
- ▶ Classification of standing waves and computations of the bifurcation diagram on tadpole graphs by C.Cacciapuoti, D.Finco, D.Noja, Phys. Rev. E **91**, 013206 (2015); rigorous results on existence, bifurcations, and stability by D.Noja, D.P., and G.Shaikhova, Nonlinearity (2015).

## NLS on a tadpole graph



NLS on a tadpole graph

$$i \frac{\partial}{\partial t} \Psi = \Delta \Psi + (p+1) |\Psi|^{2p} \Psi, \quad \Psi \in \mathcal{D}(\Delta),$$

where  $p > 0$  is the parameter for the power nonlinearity. The power nonlinearity is to be defined "edge by edge".

This is an example of interaction between NLS dynamics on a bounded and unbounded sets. Although it is special, it highlights interesting behavior.

### Problems:

- ▶ Existence and bifurcations of standing wave solutions  $\Psi = \Phi(x)e^{i\omega t}$

$$-\Delta \Phi - (p+1) |\Phi|^{2p} \Phi = \omega \Phi \quad \omega \in \mathbb{R}, \quad \Phi \in \mathcal{D}(\Delta).$$

- ▶ Spectral and orbital stability of standing waves.

## Existence of Standing waves

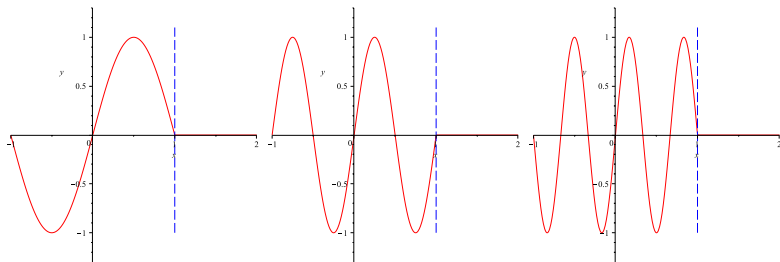
$$\begin{cases} -u''(x) - (p+1)|u|^{2p}u = \omega u, & x \in (-L, L), \\ -v''(y) - (p+1)|v|^{2p}v = \omega v, & y \in (0, \infty), \\ u(L) = u(-L) = v(0), \\ u'(L) - u'(-L) = v'(0). \end{cases}$$

### Linear spectrum:

- ▶ Essential spectrum:  $\sigma_{ess}(-\Delta) = [0, \infty)$  with resonance at 0.
- ▶ Embedded eigenvalues:  $\left\{ \lambda_n = \left(\frac{n\pi}{L}\right)^2, n \in \mathbb{N} \right\} \subset \sigma_{ess}(-\Delta)$

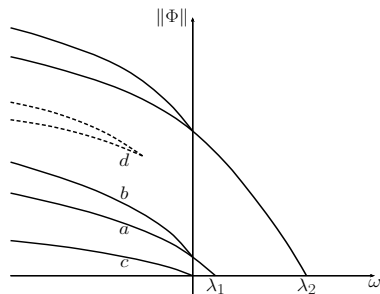
The corresponding (normalized) eigenfunctions are:

$$\Upsilon_n = \frac{1}{\sqrt{L}} \left( \sin\left(\frac{n\pi x}{L}\right), 0 \right) \quad n = 1, 2, 3, \dots$$



## Existence of Standing waves

The following bifurcation diagram has been computed for  $p = 1$  (Cacciapuoti *et al.*, 2015):



The diagram describes the families of stationary states and their possible relation with the spectrum of  $-\Delta$ .

The model, although simple, exhibits a surprisingly rich behavior

- ▶ branches of standing waves bifurcating from the embedded eigenvalues
- ▶ pitchfork bifurcation at threshold  $\omega = 0$ : edge solitons
- ▶ saddle-node bifurcations of standing waves (dashed lines)

# Standing waves bifurcating from the embedded eigenvalues

Invariant reduction

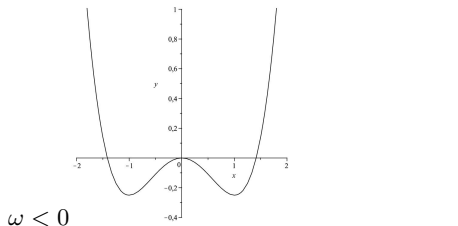
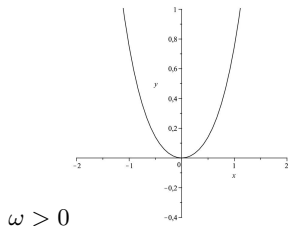
$$\begin{cases} -u''(x) - (p+1)|u|^{2p}u = \omega u, & x \in (-L, L), \\ u(L) = u(-L) = 0, \\ u'(L) = u'(-L). \end{cases}$$

Associated energy invariant

$$E = \left(\frac{du}{dx}\right)^2 + (\omega + |u|^{2p})u^2 = \text{const.}$$

For a given  $L$ , there exist two solutions  $u_{n,\omega}^{\pm}$  in  $H_{\text{per,odd}}^2(-L, L)$ ,  $n \in \mathbb{N}$  for every  $\omega \in (-\infty, \lambda_n)$ , where  $\lambda_n := \left(\frac{n\pi}{L}\right)^2$ . The map  $\omega \mapsto u_{n,\omega}^{\pm}$  is  $C^1$  in  $\omega$ .

Depending on the sign of  $\omega$  the effective potential  $V$  has two different forms.





## Numerical solutions for $p = 1$

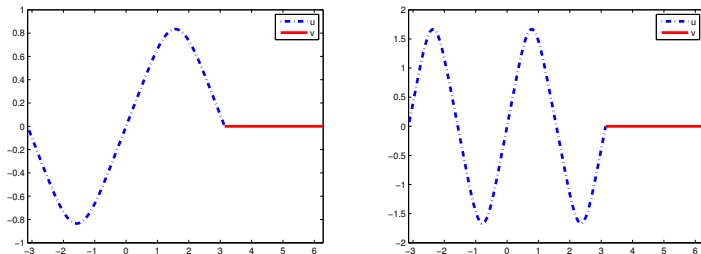


Figure : Standing wave solutions  $(u_{n,\omega}^+, 0)$  versus  $x$  for  $n = 1$  (a) and  $n = 2$  (b) corresponding to  $\omega = -1$ .

## Standing waves bifurcating from the zero resonance

Let  $\omega = -\epsilon^2$  and consider small values of  $\epsilon$ . For the solution on the tail of the tadpole, we can scale

$$v(x) = \epsilon^{\frac{1}{p}} \phi(z), \quad z = \epsilon y,$$

where  $\phi$  is a decaying solution of the second-order equation

$$-\phi''(z) + \phi - (p+1)|\phi|^{2p}\phi = 0, \quad z > 0.$$

Let  $\phi_0(z) = \operatorname{sech}^{\frac{1}{p}}(pz)$  be the unique symmetric solitary wave. Then,  $\phi(z) = \phi_0(z+a)$  for unknown parameter  $a$ .

**Bifurcation problem:**

$$\begin{cases} -u''(x) + \epsilon^2 u - (p+1)|u|^{2p}u = 0, & x \in (-L, L), \\ u(L) = u(-L) = \epsilon^{\frac{1}{p}} \phi_0(a), \\ u'(L) - u'(-L) = \epsilon^{1+\frac{1}{p}} \phi_0'(a). \end{cases}$$

- ▶ Primary branch (positive definite) bifurcating from zero solution.
- ▶ Higher branches (sign-indefinite) bifurcating from solutions  $u_{n,\omega}^{\pm}$ .

## Primary branch

Using the scaling transformation

$$u(x) = \epsilon^{\frac{1}{p}} \psi(z), \quad z = \epsilon x,$$

we can write the bifurcation problem as

$$\begin{cases} -\psi''(z) + \psi - (p+1)|\psi|^{2p}\psi = 0, & z \in (-\epsilon L, \epsilon L), \\ \psi(\epsilon L) = \psi(-\epsilon L) = \phi_0(a), \\ \psi'(\epsilon L) - \psi'(-\epsilon L) = \phi_0'(a). \end{cases}$$

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### Construction of positive solution:

- ▶ An even solution is defined near the origin:  $\psi(0) = \psi_0$ ,  $\psi'(0) = 0$ .
- ▶ The continuity boundary condition

$$\phi_0(a) = \psi(\epsilon L) = \psi_0 + \frac{1}{2}\psi''(0)\epsilon^2 L^2 + \mathcal{O}(\epsilon^4),$$

hence  $\psi_0 = \phi_0(a) + \mathcal{O}(\epsilon^2)$  is uniquely defined for every  $a \in \mathbb{R}$ .

- ▶ The flux boundary condition

$$\phi'_0(a) = 2\psi'(\epsilon L) = 2\psi''(0)\epsilon L + \mathcal{O}(\epsilon^3).$$

where  $\phi'_0(0) = 0$  and  $\phi''_0(0) \neq 0$ . Hence,  $a = 2\epsilon L + \mathcal{O}(\epsilon^3)$  is unique.

- ▶ The small solution is unique and positive:  $\psi = 1 + \mathcal{O}_{C^\infty(-L, L)}(\epsilon^2)$ .

## Higher branches

Bifurcation problem:

$$\begin{cases} -u''(x) + \epsilon^2 u - (p+1)|u|^{2p}u = 0, & x \in (-L, L), \\ u(L) = u(-L) = \epsilon^{\frac{1}{p}} \phi_0(a), \\ u'(L) - u'(-L) = \epsilon^{1+\frac{1}{p}} \phi_0'(a). \end{cases} \quad (1)$$

Since  $u \approx u_{n,\omega=-\epsilon^2}^\pm \neq 0$ , no scaling transformation can be used.

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Since  $u \approx u_{n,\omega=-\epsilon^2}^\pm \neq 0$ , no scaling transformation can be used.

Construction of particular solutions:

- ▶ Take  $u_\epsilon := u_{n,\omega=-\epsilon^2}^\pm$  and translate as  $u(x) = u_\epsilon(x+b)$  for  $b \in \mathbb{R}$ .
- ▶ Since  $u_\epsilon$  is  $2L$ -periodic, the flux boundary condition is satisfied if  $a = 0$ .
- ▶ The continuity boundary conditions yields

$$u_\epsilon(L+b) = \epsilon^{\frac{1}{p}}, \quad \phi_0(0) = 1.$$

Since  $u_\epsilon(L) = 0$  and  $u'_\epsilon(L) \neq 0$ ,  $b = \frac{1}{u'_\epsilon(L)} \epsilon^{\frac{1}{p}} + \mathcal{O}\left(\epsilon^{\frac{3}{p}}\right)$  is unique.

- ▶ Pitchfork bifurcation: two different solutions exist for  $u'_\epsilon(L) \geq 0$ .
- ▶ Uniqueness of solution  $u \approx u_\epsilon$  can be shown.

## Numerical solutions for $p = 1$

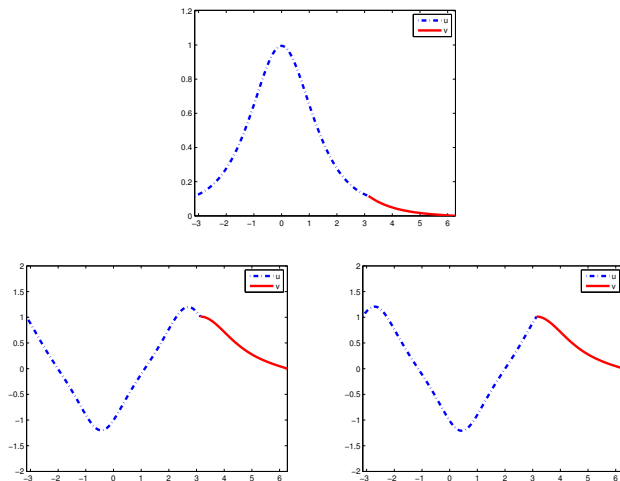
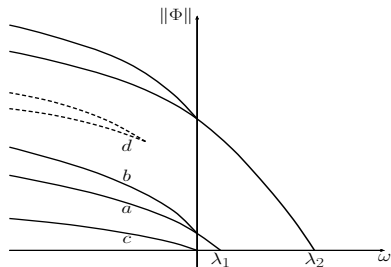


Figure : Standing wave solutions  $(u, v)$  versus  $x$  for  $\omega = -1$  along the primary branch (a) and the higher branches for  $n = 1$  (b,c).

## Summary on existence of standing waves

1. A continuous branch bifurcating from the zero energy resonance of the system
2. A continuous branch of edge solitons displaying a pitchfork bifurcation at the threshold of the continuous spectrum



### Open problems:

- ▶ Continuation of new branches in  $\omega$  to  $\omega \rightarrow -\infty$
- ▶ Prediction of saddle-node bifurcations for dashed branches.



## Orbital stability

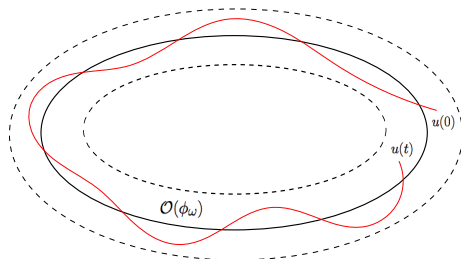
Recall that NLS has  $U(1)$ , or phase, symmetry. No stability of equilibrium *points*  $\Phi$  can hold, but stability of equilibrium *orbits*  $\{e^{i\theta}\Phi\}_{\theta \in \mathbb{R}}$  may be attained in some cases.

### Definition

We say that  $e^{i\omega t}\Phi$  is *orbitally stable* in a Banach space  $V$  if  $\forall \epsilon > 0 \exists \delta > 0$  such that  $\forall u_0 \in V$  with  $\|u_0 - \Phi\|_V < \delta$ , NLS has a global solution  $u(t) \in V$  with initial datum  $u_0$  satisfying

$$\inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta}\Phi\|_V < \epsilon,$$

for every  $t \in \mathbb{R}$ .



## Spectral stability

Linearization of NLS with  $\Psi = e^{i\omega t} (\Phi(x) + U(x, t) + iW(x, t))$  and the separation of variables  $U(x, t) = \tilde{U}(x)e^{\lambda t}$  results in the spectral problem

$$L_+ \tilde{U} = -\lambda \tilde{W}, \quad L_- \tilde{W} = \lambda \tilde{U},$$

associated with the Schrödinger operators  $L_+$  and  $L_-$ . For the tadpole graph, this yields two self-adjoint problems

$$L_- : \begin{cases} -U''(x) - \omega U - (p+1)|u|^{2p}U = \lambda U, & x \in (-L, L), \\ -V''(y) - \omega V - (p+1)|v|^{2p}V = \lambda V, & y \in (0, \infty), \\ U(L) = U(-L) = V(0), \\ U'(L) - U'(-L) = V'(0), \end{cases}$$

and

$$L_+ : \begin{cases} -U''(x) - \omega U - (2p+1)(p+1)|u|^{2p}U = \lambda U, & x \in (-L, L), \\ -V''(y) - \omega V - (2p+1)(p+1)|v|^{2p}V = \lambda V, & y \in (0, \infty), \\ U(L) = U(-L) = V(0), \\ U'(L) - U'(-L) = V'(0). \end{cases}$$

### Definition

We say that the standing wave is spectrally unstable if there exist an eigenvector  $\tilde{U}, \tilde{W} \in \mathcal{D}(-\Delta)$  for an eigenvalue with  $\operatorname{Re}(\lambda) > 0$ . Otherwise, we say that it is weakly spectrally stable.

## Criteria for spectral stability

Constrained space associated with the  $U(1)$ , or phase, symmetry:

$$L_c^2 := \{U \in L^2 : \langle U, \Phi \rangle_{L^2} = 0\}.$$

Denote the number of negative eigenvalues of  $L_{\pm}$  by  $n(L_{\pm})$  and assume that  $L_+$  is invertible and  $\text{Ker}(L_-)$  is one-dimensional.

The following criteria summarize the results from Shatah–Straus (1983), Weinstein (1985), Grillakis (1990), Jones (1990), Kapitula–Kevrekidis–Stanstede (2004), P. (2005), etc.

- ▶ If  $n(L_+) = 1$  and  $n(L_-) = 0$ , then
  - ▶  $\Phi$  is spectrally and orbitally stable if  $n(L_+|_{L_c^2}) = 0$
  - ▶  $\Phi$  is unstable if  $n(L_+|_{L_c^2}) = 1$ .
- ▶ If  $n(L_+|_{L_c^2}) - n(L_-)$  is nonzero, then  $\Phi$  is unstable.
- ▶ If  $n(L_+|_{L_c^2}) + n(L_-)$  is odd, then  $\Phi$  is unstable.
- ▶ If  $n(L_+|_{L_c^2}) + n(L_-)$  is even, then  $\Phi$  is stable if there exist  $n(L_+|_{L_c^2}) + n(L_-)$  eigenvalues  $\lambda \in i\mathbb{R}$  of negative Krein signature.

## Stability of the primary branch for $\omega = -\epsilon^2$

Recall that

$$\Phi = \epsilon^{\frac{1}{p}}(\psi(z), \phi(z)), \quad z = \epsilon x,$$

where  $\psi(z) = 1 + \mathcal{O}(z^2)$  and  $\phi(z) = \phi_0(z + a)$ .

- ▶  $n(L_-) = 0$  because  $\Phi$  is strictly positive.
- ▶  $n(L_+) = 1$  because of the reduction to the scalar Schrödinger equation.

The spectral problem for  $L_+$  with  $\lambda = \epsilon^2 \Lambda$  is

$$\begin{cases} -U''(z) + U(z) - (2p+1)(p+1)|\psi(z)|^{2p}U(z) = \Lambda U(z), & z \in (-\epsilon L, \epsilon L), \\ -V''(z) + V(z) - (2p+1)(p+1)|\phi(z)|^{2p}V(z) = \Lambda V(z), & z \in (0, \infty), \\ U(\epsilon L) = U(-\epsilon L) = V(0), \\ U'(\epsilon L) - U'(-\epsilon L) = V'(0), \end{cases}$$

The leading-order spectral problem is related to the scalar Schrödinger equation on the line

$$\begin{cases} -V''(z) + V(z) - (2p+1)(p+1)\operatorname{sech}^2(pz)V(z) = \Lambda V(z), & z \in (0, \infty), \\ V'(0) = 0, \end{cases}$$

which has only one negative eigenvalue  $\Lambda_0 < 0$ .

## Stability of the primary branch

- ▶  $n(L_+|_{L^2}) = 0$  if the slope condition is satisfied

$$\frac{d}{d\omega} \|\Phi\|^2 < 0$$

This can be checked directly from asymptotic solutions:

$$\|u\|_{L^2(-L,L)}^2 = \epsilon^{\frac{2}{p}} \|\psi(\epsilon \cdot)\|_{L^2(-L,L)}^2 = \epsilon^{\frac{2}{p}} (2L + \mathcal{O}(\epsilon^2))$$

and

$$\|v\|_{L^2(0,\infty)}^2 = \epsilon^{\frac{2}{p}-1} \|\phi_0\|_{L^2(a,\infty)}^2 = \epsilon^{\frac{2}{p}-1} (\|\phi_0\|_{L^2(0,\infty)}^2 + \mathcal{O}(\epsilon)).$$

### Theorem

*For  $\omega = -\epsilon^2$  with  $\epsilon > 0$  sufficiently small, the primary branch is orbitally stable for every  $p \in (0, 2)$  and orbitally unstable for every  $p \in (2, \infty)$ .*

### Open problems:

- ▶ For  $p = 1$ , show that  $\Phi$  is a constrained minimizer of  $S_\omega$ ,  $\omega < 0$ .
- ▶ For  $p = 2$ , the test for orbital stability is inconclusive.

## Stability of the higher branches for $\omega = -\epsilon^2$

The degenerate higher branches are:

$$\Phi = (u_{n,\omega}^{\pm}(x+b), 0).$$

Numerical solutions for  $p = 1$

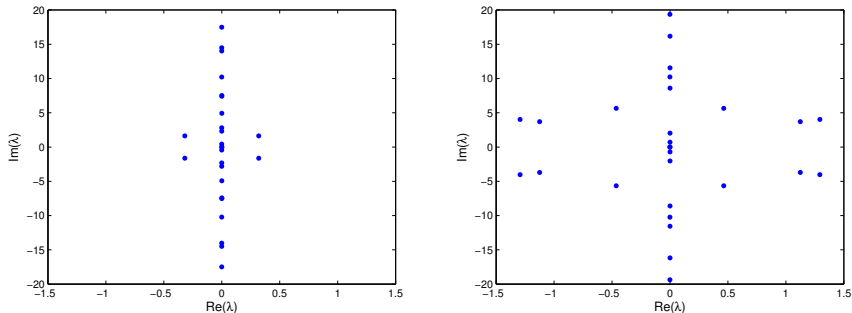


Figure : Eigenvalues  $\lambda$  of the spectral stability problem on the complex plane for the standing wave solutions  $(u_{n,\omega}^{\pm}, 0)$  with  $n = 1$  (a) and  $n = 2$  (b) for  $\omega = -1$ .

## Conclusions and Perspectives

- The classification of nonlinear bound states for the cubic NLS equation on a tadpole graph exhibits behavior previously unknown for the standard NLS equation with power nonlinearity on the line.
- The analysis suggests soliton bifurcations from the edge of the continuum spectrum is a general feature when stationary states on a bounded interval are coupled with stationary states on the unbounded interval. Stability properties are accessible near the bifurcation.
- Complete the stability analysis:
  - ▶ Energy minimization properties for the primary branch
  - ▶ Spectral stability along the higher branches as  $\omega \rightarrow -\infty$
- Other graphs: influence of geometry (or more complex topology)?