Existence and stability of standing waves for the NLS equation on a tadpole graph

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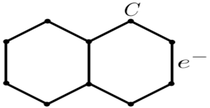
Introduction

Graph models for the dynamics of constrained quantum particles were first suggested by Pauling and then used by Ruedenberg and Scherr in 1953 to study the spectrum of aromatic hydrocarbons.

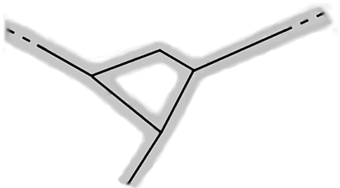


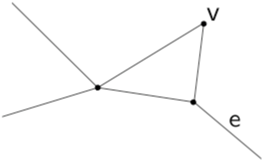
Nowadays graph models are widely used in the modeling of quantum dynamics of thin graph-like structures (quantum wires, nanotechnology, large molecules, periodic arrays in solids, photonic crystals...).

- G. Berkolaiko, P. Kuchment, *Introduction to Quantum Graphs* (AMS, Providence, 2013).
- P. Exner, J. Keating, P. Kuchment, T. Sunada, and A. Teplyaev, *Analysis on graphs and its applications*, Proceedings of Symposia in Pure Mathematics, AMS 2008.

Metric Graphs

Graphs are one-dimensional approximations for constrained dynamics in which transverse dimensions are small with respect to longitudinal ones.



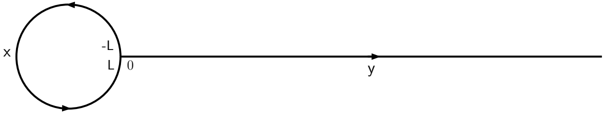


A metric graph is realized by a set of edges and vertices, with a metric structure on each edge. Proper boundary conditions are needed on the vertices to ensure that certain differential operators defined on graphs are self-adjoint.

Kirchhoff boundary conditions:

- Functions in each edge have the same value at each vertex.
- Sum of fluxes (signed derivatives of functions) is zero at each vertex.

Tadpole Graph



The ring is placed on the interval $[-L, L]$ and the semi-infinite interval is $[0, \infty)$. The Laplacian operator is defined by

$$\Delta \Psi = \left[\begin{array}{ll} u''(x), & x \in (-L, L) \\ v''(y), & y \in (0, \infty) \end{array} \right],$$

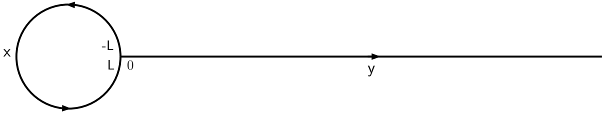
acting on functions in the form

$$\Psi = \left[\begin{array}{ll} u(x), & x \in (-L, L) \\ v(y), & y \in (0, \infty) \end{array} \right],$$

in the domain

$$\mathcal{D}(\Delta) = \left\{ \begin{array}{l} (u, v) \in H^2(-L, L) \times H^2(0, \infty) : \\ u(L) = u(-L) = v(0), \quad u'(L) - u'(-L) = v'(0) \end{array} \right\}.$$

Laplacian on a Tadpole Graph



The Kirchhoff boundary conditions are symmetric:

$$\langle u_1, u_2'' \rangle_{L^2(-L,L)} + \langle v_1, v_2'' \rangle_{L^2(0,\infty)} = \langle u_1'', u_2 \rangle_{L^2(-L,L)} + \langle v_1'', v_2 \rangle_{L^2(0,\infty)}$$

Indeed, we have

$$\langle \Psi_1, \Delta \Psi_2 \rangle - \langle \Delta \Psi_1, \Psi_2 \rangle = \left[\bar{u}_1' u_2 - \bar{u}_1 u_2' \right] \Big|_{-L}^{L} + \left[\bar{v}_1' v_2 - \bar{v}_1 v_2' \right] \Big|_{0} = 0,$$

if functions decomposed as $\Psi = (u, v)$ in the "head" and "tail" of the tadpole satisfy the Kirchhoff boundary conditions:

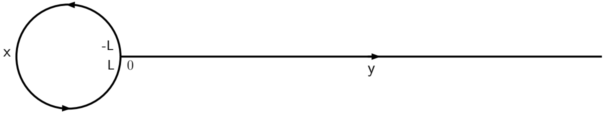
$$\left\{ \begin{array}{l} u(L) = u(-L) = v(0) \\ v'(0) - u'(L) + u'(-L) = 0 \end{array} \right.$$

NLS on star graphs

New subject with several recent results:

- Gnutzmann-Smilansky-Derevyanko, Phys. Rev. A **83** (2011), 033831: a complex set of resonances after inserting a single nonlinear edge in a linear quantum graph; recent rigorous analysis by L.Tentarelli, arXiv:1503.00455.
- Series of papers on star graphs by Adami-Cacciapuoti-Finco-Noja: Scattering of solitons; Standing waves and stability (2011-14).
- Recent work by Adami-Serra-Tilli on nonexistence of ground states in networks with closed cycles, arXiv:1406.4036.
- Results on dispersive estimates on trees (including star graphs) in V.Banica-L.Ignat (2011-2014).
- Classification of standing waves and computations of the bifurcation diagram on tadpole graphs by C.Cacciapuoti, D.Finco, D.Noja, Phys. Rev. E **91**, 013206 (2015); rigorous results on existence, bifurcations, and stability by D.Noja, D.P., and G.Shaikhova, arXiv:1412.8232.

NLS on a tadpole graph



NLS on a tadpole graph

$$i\frac{\partial}{\partial t}\Psi = \Delta\Psi + (p+1)|\Psi|^{2p}\Psi, \quad \Psi \in \mathcal{D}(\Delta),$$

where $p > 0$ is the parameter for the power nonlinearity. The power nonlinearity is to be defined "edge by edge".

This is an example of interaction between NLS dynamics on a bounded and unbounded sets. Although it is special, it highlights interesting and general behaviors.

Problems:

- Existence and bifurcations of standing wave solutions $\Psi = \Phi(x)e^{i\omega t}$

$$-\Delta\Phi - (p+1)|\Phi|^{2p}\Phi = \omega\Phi \qquad \omega \in \mathbb{R}\,, \ \Phi \in \mathcal{D}(\Delta).$$

- Spectral and orbital stability of standing waves.

Existence of Standing waves

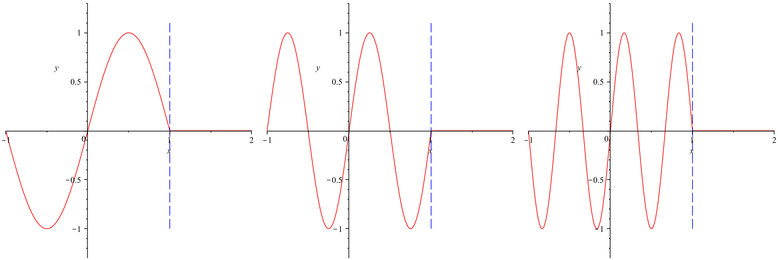
$$\begin{cases} -u''(x) - (p+1)|u|^{2p}u = \omega u, & x \in (-L, L)\ , \\ -v''(y) - (p+1)|v|^{2p}v = \omega v, & y \in (0, \infty)\ , \\ u(L) = u(-L) = v(0)\ , \\ u'(L) - u'(-L) = v'(0)\ . \end{cases}$$

Linear spectrum:

- Essential spectrum: $\sigma_{ess}(-\Delta) = [0, \infty)$ with resonance at 0.
- Embedded eigenvalues: $\left\{\lambda_n = \left(\frac{n\pi}{L}\right)^2,\ n \in \mathbb{N}\right\} \subset \sigma_{ess}(-\Delta)$

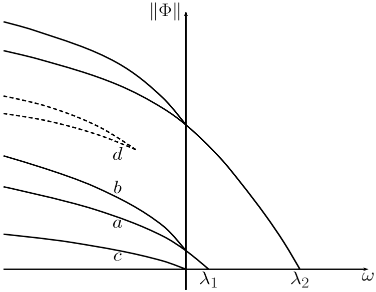
  The corresponding (normalized) eigenfunctions are:

$$\Upsilon_n = \frac{1}{\sqrt{L}}\left(\sin\left(\frac{n\pi x}{L}\right), 0\right) \qquad n = 1, 2, 3, \ldots$$



Existence of Standing waves

The following bifurcation diagram has been computed for $p = 1$ (Cacciapuoti *et al.*, 2015):



The diagram describes the families of stationary states and their possible relation with the spectrum of $-\Delta$.

The model, although simple, exhibits a surprisingly rich behavior

- branches of standing waves bifurcating from the embedded eigenvalues
- pitchfork bifurcation at threshold $\omega = 0$: edge solitons
- branches of non linearly related standing waves (dashed lines)

Standing waves bifurcating from the embedded eigenvalues

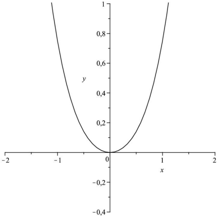
Invariant reduction

$$\begin{cases} -u''(x) - (p+1)|u|^{2p}u = \omega u, & x \in (-L, L), \\ u(L) = u(-L) = 0, \\ u'(L) = u'(-L). \end{cases}$$

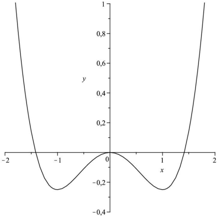
Associated energy invariant

$$E = \left(\frac{du}{dx}\right)^2 + \left(\omega + |u|^{2p}\right) u^2 = \text{const.}$$

Depending on the sign of ω the effective potential V has two different forms.



$\omega > 0$



$\omega < 0$

Lemma

For every $\omega \geq 0$, the period-to-energy map

$$\left(0, \frac{\pi}{\sqrt{\omega}}\right) \ni T \to E \in (0, \infty)$$

associated with the closed periodic trajectories that surrounds the zero critical point of the energy invariant is a C^1 diffeomorphism with

$$E'(T) < 0, \quad \text{for all } \; T \in \left(0, \frac{\pi}{\sqrt{\omega}}\right).$$

For every $\omega < 0$, the period-to-energy map

$$(0, \infty) \ni T \to E \in (0, \infty)$$

associated with the closed periodic trajectories that surround all critical points of the energy invariant is a C^1 diffeomorphism with

$$E'(T) < 0, \quad \text{for all } \; T \in (0, \infty).$$

Standing waves bifurcating from the embedded eigenvalues

Proposition

The existence problem with $\omega \in (-\infty, \lambda_n)$ *admits two solutions* $u^{\pm}_{n,\omega}$ *in* $H^2_{\mathrm{per,odd}}(-L, L)$, $n \in \mathbb{N}$. *The map* $\omega \mapsto u^{\pm}_{n,\omega} \in H^2_{\mathrm{per,odd}}(-L, L)$ *is* C^1 *in* ω.

Standing waves bifurcating from the embedded eigenvalues

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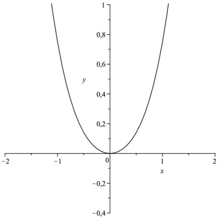
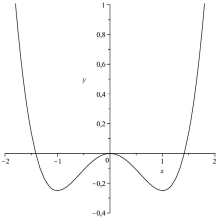
Steps:

- Two C^1 solution branches $u^{\pm}_{n,\omega}$ bifurcate from $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and exist for $\omega \lesssim \lambda_n$. By symmetry, $u^{-}_{n,\omega}(x) = u^{+}_{n,\omega}(-x)$.
- If $\omega \in (0, \lambda_n)$, then $T_n = L/n$ belongs to the range $\left(0, \frac{\pi}{\sqrt{\omega}}\right)$ and the two C^1 solution branches correspond to the energy level $E_n = E(T_n)$.
- If $\omega \leq 0$, the two C^1 solution branches continue uniquely for the same energy level E_n.



$\omega > 0$



$\omega < 0$

Explicit solutions for $p = 1$

Solutions are explicitly constructed by using Jacobian elliptic functions:

$$u_{cn}(x;k) = \sqrt{\frac{\omega k^2}{1-2k^2}}\,\mathrm{cn}\left(\sqrt{\frac{\omega}{1-2k^2}}x, k\right) \qquad \omega \in \mathbb{R}\,;\; k \in (0,1),$$

with period

$$T_{cn}(k) = 4\sqrt{(1-2k^2)/\omega}\,K(k),$$

where K is the Legendre's complete elliptic integral of the first kind

$$K(\gamma) = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-\gamma^2 t^2)}}dt$$

The condition

$$2L = n\,T_{cn}(k) \qquad n \in \mathbb{N}$$

fixes $k_n(\omega)$.

- As $k \to 0$, we have $K(k) \to \frac{\pi}{2}$ and $\omega \to \lambda_n = \left(\frac{n\pi}{L}\right)^2$.
- As $k \to 1$, we have $K(k) \to \infty$ and $\omega \to -\infty$.

Numerical solutions for $p = 1$

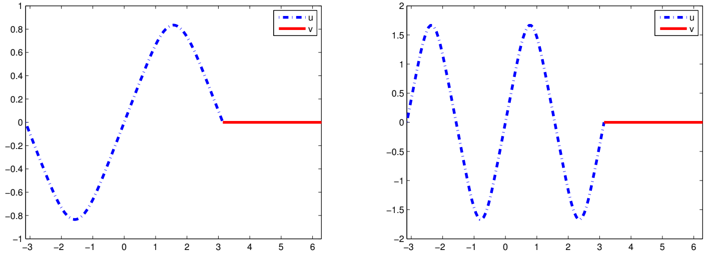


Figure : Standing wave solutions $(u^{+}_{n,\omega}, 0)$ versus x for $n = 1$ (a) and $n = 2$ (b) corresponding to $\omega = -1$.

Standing waves bifurcating from the zero resonance

Let $\omega = -\epsilon^2$ and consider small values of ϵ. For the solution on the tail of the tadpole, we can scale

$$v(x) = \epsilon^{\frac{1}{p}} \phi(z), \quad z = \epsilon y,$$

where ϕ is a decaying solution of the second-order equation

$$-\phi''(z) + \phi - (p+1)|\phi|^{2p}\phi = 0, \quad z > 0.$$

Let $\phi_0(z) = \operatorname{sech}^{\frac{1}{p}}(pz)$ be the unique symmetric solitary wave. Then, $\phi(z) = \phi_0(z+a)$ for unknown parameter a.

Bifurcation problem:

$$\left\{ \begin{array}{l} -u''(x) + \epsilon^2 u - (p+1)|u|^{2p} u = 0, \quad x \in (-L, L), \\ u(L) = u(-L) = \epsilon^{\frac{1}{p}} \phi_0(a), \\ u'(L) - u'(-L) = \epsilon^{1+\frac{1}{p}} \phi_0'(a). \end{array} \right. \qquad (1)$$

- Primary branch (positive definite) bifurcating from zero solution.
- Higher branches (sign-indefinite) bifurcating from solutions $u^{\pm}_{n,\omega}$.

Primary branch

Using the scaling transformation

$$u(x) = \epsilon^{\frac{1}{p}} \psi(z), \quad z = \epsilon x,$$

we can write the bifurcation problem as

$$\begin{cases} -\psi''(z) + \psi - (p+1)|\psi|^{2p}\psi = 0, & z \in (-\epsilon L, \epsilon L), \\ \psi(\epsilon L) = \psi(-\epsilon L) = \phi_0(a), \\ \psi'(\epsilon L) - \psi'(-\epsilon L) = \phi_0'(a). \end{cases}$$

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Construction of positive solution:

- An even solution is defined near the origin: $\psi(0) = \psi_0$, $\psi'(0) = 0$.
- The continuity boundary condition

$$\phi_0(a) = \psi(\epsilon L) = \psi_0 + \frac{1}{2}\psi''(0)\epsilon^2 L^2 + \mathcal{O}(\epsilon^4),$$

  hence $\psi_0 = \phi_0(a) + \mathcal{O}(\epsilon^2)$ is uniquely defined for every $a \in \mathbb{R}$.
- The flux boundary condition

$$\phi_0'(a) = 2\psi'(\epsilon L) = 2\psi''(0)\epsilon L + \mathcal{O}(\epsilon^3).$$

  where $\phi_0'(0) = 0$ and $\phi_0''(0) \neq 0$. Hence, $a = 2\epsilon L + \mathcal{O}(\epsilon^3)$ is unique.
- The small solution is unique and positive: $u = \epsilon^{\frac{1}{p}}(1 + \mathcal{O}_{C^\infty(-L,L)}(\epsilon^2))$.

Higher branches

Bifurcation problem:

$$\left\{\begin{array}{l} -u''(x) + \epsilon^2 u - (p+1)|u|^{2p} u = 0, \quad x \in (-L, L), \\ u(L) = u(-L) = \epsilon^{\frac{1}{p}} \phi_0(a), \\ u'(L) - u'(-L) = \epsilon^{1+\frac{1}{p}} \phi_0'(a). \end{array}\right. \tag{2}$$

Since $u \approx u^{\pm}_{n,\omega=-\epsilon^2} \neq 0$, no scaling transformation can be used.

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Construction of particular solutions:

- Take $u_\epsilon := u^{\pm}_{n,\omega=-\epsilon^2}$ and translate as $u(x) = u_\epsilon(x+b)$ for $b \in \mathbb{R}$.
- Since u_ϵ is $2L$-periodic, the flux boundary condition is satisfied if $a = 0$.
- The continuity boundary conditions yields

$$u_\epsilon(L+b) = \epsilon^{\frac{1}{p}}, \quad \phi_0(0) = 1.$$

  Since $u_\epsilon(L) = 0$ and $u_\epsilon'(L) \neq 0$, $b = \frac{1}{u_\epsilon'(L)} \epsilon^{\frac{1}{p}} + \mathcal{O}\left(\epsilon^{\frac{3}{p}}\right)$ is unique.
- Pitchfork bifurcation: two different solutions exist for $u_\epsilon'(L) \gtrless 0$.

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- Pitchfork bifurcation: two different solutions exist for $u_\epsilon'(L) \gtrless 0$.

Uniqueness of solution $u \approx u_\epsilon$ can be shown.

Numerical solutions for $p = 1$

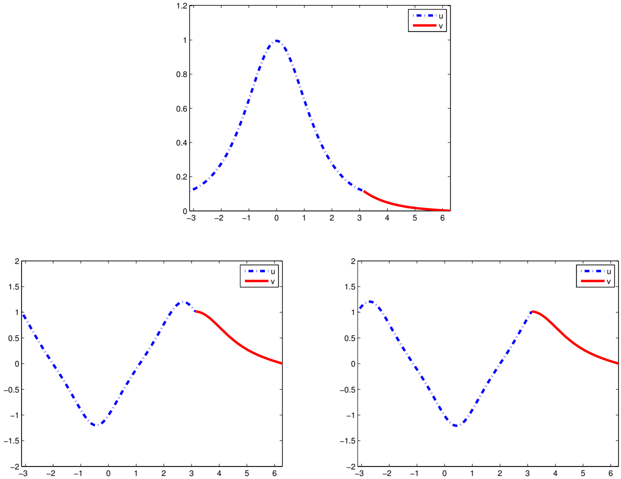
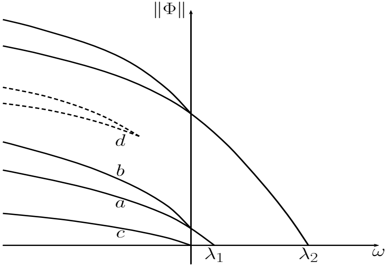


Figure : Standing wave solutions (u, v) versus x for $\omega = -1$ along the primary branch (a) and the higher branches for $n = 1$ (b,c).

Existence of Standing waves

1. A continuous branch bifurcating from the zero energy resonance of the system
2. A continuous branch of edge solitons displaying a pitchfork bifurcation at the threshold of the continuous spectrum



Open problems:

- Continuation of new branches in ω to $\omega \to -\infty$
- Prediction of saddle-node bifurcations observed for the dashed branches.

Orbital stability

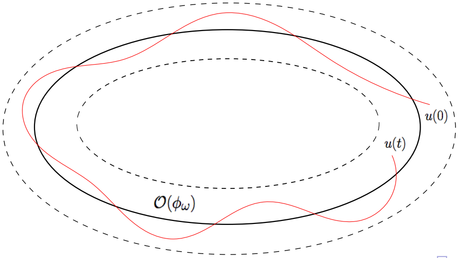
Recall that NLS has $U(1)$, or phase, symmetry. No stability of equilibrium *points* Φ can hold, but stability of equilibrium *orbits* $\left\{e^{i\theta}\Phi\right\}_{\theta\in\mathbb{R}}$ may be attained in some cases.

Definition

We say that $e^{i\omega t}\Phi$ is *orbitally stable* in a Banach space V if $\forall\epsilon > 0\ \exists\delta > 0$ such that $\forall u_0 \in V$ with $\|u_0 - \Phi\|_V < \delta$, NLS has a global solution $u(t) \in V$ with initial datum u_0 satisfying

$$\inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\Phi\|_V < \epsilon,$$

for every $t \in \mathbb{R}$.



Spectral stability

Linearization of NLS with $\Psi = e^{i\omega t}(\Phi(x) + U(x,t) + iW(x,t))$ and the separation of variables $U(x,t) = \tilde{U}(x)e^{\lambda t}$ results in the spectral problem

$$L_+\tilde{U} = -\lambda\tilde{W}, \quad L_-\tilde{W} = \lambda\tilde{U},$$

associated with the Schrödinger operators L_+ and L_-. For the tadpole graph, this yields two self-adjoint problems

$$L_- : \quad \left\{ \begin{array}{ll} -U''(x) - \omega U - (p+1)|u|^{2p}U = \lambda U, & x \in (-L, L), \\ -V''(y) - \omega V - (p+1)|v|^{2p}V = \lambda V, & y \in (0, \infty), \\ U(L) = U(-L) = V(0), & \\ U'(L) - U'(-L) = V'(0), & \end{array} \right.$$

and

$$L_+ : \quad \left\{ \begin{array}{ll} -U''(x) - \omega U - (2p+1)(p+1)|u|^{2p}U = \lambda U, & x \in (-L, L), \\ -V''(y) - \omega V - (2p+1)(p+1)|v|^{2p}V = \lambda V, & y \in (0, \infty), \\ U(L) = U(-L) = V(0), & \\ U'(L) - U'(-L) = V'(0). & \end{array} \right.$$

Definition

We say that the standing wave is spectrally unstable if there exist an eigenvector $\tilde{U}, \tilde{W} \in \mathcal{D}(-\Delta)$ for an eigenvalue with $\mathrm{Re}(\lambda) > 0$. Otherwise, we say that it is weakly spectrally stable.

Criteria for spectral stability

Constrained space associated with the $U(1)$, or phase, symmetry:

$$L_c^2 := \left\{U \in L^2 : \quad \langle U, \Phi \rangle_{L^2} = 0\right\}.$$

Denote the number of negative eigenvalues of $L_\pm$ by $n(L_\pm)$ and assume that L_+ is invertible and $\mathrm{Ker}(L_-)$ is one-dimensional.

The following criteria summarize the results from Shatah–Straus (1983), Weinstein (1985), Grillakis (1990), Jones (1990), Kapitula–Kevrekidis–Stanstede (2004), P. (2005), etc.

- If $n(L_+) = 1$ and $n(L_-) = 0$, then
  - Φ is stable if $n(L_+|_{L_c^2}) = 0$
  - Φ is unstable if $n(L_+|_{L_c^2}) = 1$.
- If $n(L_+|_{L_c^2}) - n(L_-)$ is nonzero, then Φ is unstable.
- If $n(L_+|_{L_c^2}) + n(L_-)$ is odd, then Φ is unstable.
- If $n(L_+|_{L_c^2}) + n(L_-)$ is even, then Φ is stable if there exist $n(L_+|_{L_c^2}) + n(L_-)$ eigenvalues $\lambda \in i\mathbb{R}$ of negative Krein signature.

Criteria for orbital stability

Orbital stability in H^1 can be proven for the case $n(L_+|_{L^2_c}) + n(L_-) = 0$. One can construct a Lyapunov function in H^1:

$$S_\omega(\Psi) = E(\Psi) + \omega\, |\Psi|^2_{L^2}\,,$$

where $E(\Psi)$ is the Hamiltonian of the NLS. Then, $S'_\omega(\Psi) = 0$ is just the stationary NLS equation with the solution $\Psi = \Phi$. The Hessian operator for $S''_\omega(\Phi)$ is diagonalized by the Schrödinger operators L_+ and L_-.

Strict coercivity holds in $L^2_c \cap H^1$:

$$\langle L_+ U, U\rangle_{L^2} \geq C_+ \|U\|^2_{H^1}, \qquad \langle L_- W, W\rangle_{L^2} \geq C_- \|W\|^2_{H^1},$$

for some $C_\pm > 0$.

For every ϵ, if δ is small enough, one has

$$\begin{aligned}
\epsilon^2 &> S_\omega(\Psi_0) - S_\omega(\Phi) = S_\omega(\Psi(t)) - S_\omega(\Phi) \\
&= S_\omega(e^{i\theta}\Psi(t)) - S_\omega(\Phi) = S_\omega(\Phi + U(t) + iW(t))) - S_\omega(\Phi) \\
&= \langle L_+ U, U\rangle_{L^2} + \langle L_- W, W\rangle_{L^2} + R, \\
&\geq C_+ \|U\|^2_{H^1} + C_- \|W\|^2_{H^1} + R,
\end{aligned}$$

where the cubic remainder R is controlled in H^1 norm.

Stability of the primary branch for $\omega = -\epsilon^2$

Recall that

$$\Phi = \epsilon^{\frac{1}{p}}(\psi(z), \phi(z)), \quad z = \epsilon x,$$

where $\psi(z) = 1 + \mathcal{O}(z^2)$ and $\phi(z) = \phi_0(z + a)$.

- $n(L_-) = 0$ because Φ is strictly positive.
- $n(L_+) = 1$ because of the scaling arguments.

The spectral problem for L_+ with $\lambda = \epsilon^2 \Lambda$ is

$$\left\{ \begin{array}{ll} -U''(z) + U(z) - (2p+1)(p+1)|\psi(z)|^{2p} U(z) = \Lambda U(z), & z \in (-\epsilon L, \epsilon L), \\ -V''(z) + V(z) - (2p+1)(p+1)|\phi(z)|^{2p} V(z) = \Lambda V(z), & z \in (0, \infty), \\ U(\epsilon L) = U(-\epsilon L) = V(0), & \\ U'(\epsilon L) - U'(-\epsilon L) = V'(0), & \end{array} \right.$$

The leading-order spectral problem is related to the scalar Schrödinger equation on the line

$$\left\{ \begin{array}{ll} -V''(z) + V(z) - (2p+1)(p+1)\mathrm{sech}^2(pz) V(z) = \Lambda V(z), & z \in (0, \infty), \\ V'(0) = 0, & \end{array} \right.$$

which has only one negative eigenvalue $\Lambda_0 < 0$.

Stability of the primary branch

- $n(L_+|_{L^2_c}) = 0$ if the slope condition is satisfied

$$\frac{d}{d\omega}\|\Phi\|^2 < 0$$

This can be checked directly from asymptotic solutions:

$$\|u\|^2_{L^2(-L,L)} = \epsilon^{\frac{2}{p}}\|\psi(\epsilon\cdot)\|^2_{L^2(-L,L)} = \epsilon^{\frac{2}{p}}\left(2L + \mathcal{O}(\epsilon^2)\right)$$

and

$$\|v\|^2_{L^2(0,\infty)} = \epsilon^{\frac{2}{p}-1}\|\phi_0\|^2_{L^2(a,\infty)} = \epsilon^{\frac{2}{p}-1}\left(\|\phi_0\|^2_{L^2(0,\infty)} + \mathcal{O}(\epsilon)\right).$$

Theorem

For $\omega = -\epsilon^2$ with $\epsilon > 0$ sufficiently small, the primary branch is orbitally stable for every $p \in (0,2)$ and orbitally unstable for every $p \in (2,\infty)$.

Open problems:

- For $p = 1$, show that Φ is a constrained minimizer of S_ω, $\omega < 0$.
- For $p = 2$, the test for orbital stability is inconclusive.

Stability of the higher branches for $\omega = -\epsilon^2$

Recall that

$$\Phi = (u^{\pm}_{n,\omega}(x+b), \epsilon^{\frac{1}{p}}\phi_0(\epsilon y)).$$

Therefore, it makes sense to count negative eigenvalues of L_- and L_+ for the limiting solution $\Phi = (u^{\pm}_{n,\omega}(x), 0)$:

$$L_- : \quad \begin{cases} -U''(x) - \omega U - (p+1)|u^{\pm}_{n,\omega}|^{2p} U = \lambda U, & x \in (-L, L), \\ -V''(y) - \omega V = \lambda V, & y \in (0, \infty), \\ U(L) = U(-L) = V(0), & \\ U'(L) - U'(-L) = V'(0), & \end{cases}$$

and

$$L_+ : \quad \begin{cases} -U''(x) - \omega U - (2p+1)(p+1)|u^{\pm}_{n,\omega}|^{2p} U = \lambda U, & x \in (-L, L), \\ -V''(y) - \omega V = \lambda V, & y \in (0, \infty), \\ U(L) = U(-L) = V(0), & \\ U'(L) - U'(-L) = V'(0). & \end{cases}$$

Count of negative eigenvalues for $(u^{\pm}_{n,\omega}(x), 0)$

$$L_- : \quad \left\{ \begin{array}{ll} -U''(x) - \omega U - (p+1)|u^{\pm}_{n,\omega}|^{2p} U = \lambda U, & x \in (-L, L), \\ -V''(y) - \omega V = \lambda V, & y \in (0, \infty), \\ U(L) = U(-L) = V(0), & \\ U'(L) - U'(-L) = V'(0), & \end{array} \right.$$

- Eigenfunctions with $V \equiv 0$ and $U \in H^2_{\mathrm{per,odd}}(-L, L)$:

  Sturm theory can be used to prove $n-1$ negative eigenvalues of L_- and n negative eigenvalues of L_+ for every $\omega < \lambda_n = \left(\frac{n\pi}{L}\right)^2$.

- Eigenfunctions with $V(x) = U(L) e^{-x\sqrt{|\omega+\lambda|}}$ for $\omega + \lambda < 0$ and $U \in H^2_{\mathrm{even}}(-L, L)$ but $U \notin H^2_{\mathrm{per}}(-L, L)$ because

$$2U'(L) = -U(L)\sqrt{|\omega + \lambda|} \neq 0.$$

  Continuation from $\omega = \lambda_n$ to $\omega = 0$ can be used to show n negative eigenvalues for either L_- or L_+ for $\omega \lesssim 0$.

Numerical solutions for $p = 1$

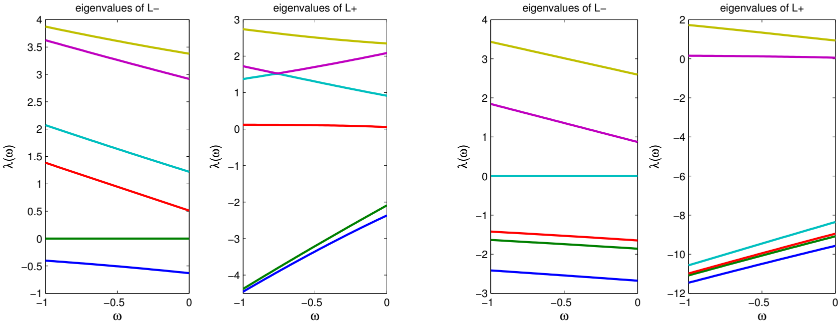


Figure : Lowest six eigenvalues of operators L_- and L_+ versus parameter ω for the standing wave solution $(u^+_{n,\omega}, 0)$ with $n = 1$ (a) and $n = 2$ (b).

Count of negative eigenvalues for $(u^{\pm}_{n,\omega}(x+b), \epsilon^{\frac{1}{p}}\phi_0(\epsilon y))$

$$L_- : \quad \begin{cases} -U''(x) + \epsilon^2 U - (p+1)|u_\epsilon(x+b)|^{2p} U = \lambda U, & x \in (-L, L), \\ -V''(y) + \epsilon^2 V - \epsilon^2 (p+1)|\phi_0(\epsilon y)|^{2p} V = \lambda V, & y \in (0, \infty), \\ U(L) = U(-L) = V(0), \\ U'(L) - U'(-L) = V'(0), \end{cases}$$

- Negative eigenvalues for the branch $(u^{\pm}_{n,\omega}(x), 0)$ persist for the branch $(u^{\pm}_{n,\omega}(x+b), \epsilon^{\frac{1}{p}}\phi_0(\epsilon y))$ if ϵ is small.

- One more (small) negative eigenvalue of operator L_+ appears if ϵ is small.

Numerical solutions for $p = 1$

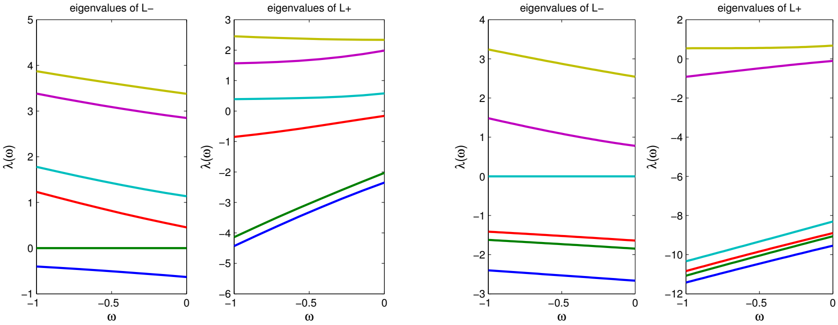


Figure : Lowest six eigenvalues of operators L_- and L_+ versus parameter ω for the standing wave solution (u, v) along the higher branches with $n = 1$ (a) and $n = 2$ (b).

Spectral instability for $(u^{\pm}_{n,\omega}(x+b), \epsilon^{\frac{1}{p}}\phi_0(\epsilon y))$

- $n(L_-) = 2n-1$
- $n(L_+) = 2n+1$
- $n(L_+|_{L^2_c}) = 2n$ if the slope condition is satisfied

$$\frac{d}{d\omega}\|\Phi\|^2 < 0$$

This can be checked directly from asymptotic solutions:

$$\|u\|^2_{L^2(-L,L)} = \|u^{\pm}_{n,\omega}\|^2_{L^2(-L,L)} = \|u^{\pm}_{n,0}\|^2_{L^2(-L,L)} + \mathcal{O}(\epsilon^2)$$

and

$$\|v\|^2_{L^2(0,\infty)} = \epsilon^{\frac{2}{p}-1}\|\phi_0\|^2_{L^2(0,\infty)}.$$

Theorem

For $\omega = -\epsilon^2$ with $\epsilon > 0$ sufficiently small, all higher branches are spectrally unstable with at least one pair (two pairs) of real eigenvalues λ in the above spectral stability problem for $p \in (0,2]$ (respectively, $p \in (2,\infty)$).

Open problems:

- Show that this conclusion remains for every $\omega < 0$.

Spectral instability for $(u^{\pm}_{n,\omega}(x), 0)$

- $n(L_-) = 2n - 1$
- $n(L_+) = 2n$
- $n(L_+|_{L^2_c}) = 2n - 1$ for $p \in (0, 2]$ because

$$\frac{d}{d\omega}\|u^{\pm}_{n,\omega}\|^2_{L^2} < 0, \quad \omega \in (-\infty, \lambda_n)$$

(Fukuizumi *et al.*, 2012)

- Spectral stability test is inconclusive.
- The spectral bands overlap on $i\mathbb{R}$ and admit no gaps for eigenvalues $\lambda \in i\mathbb{R}$ of negative Krein signature.

Open problems:

- Prove the spectral instability for small ϵ.
- Consider stability changes for large negative values of ω.

Numerical solutions for $p = 1$

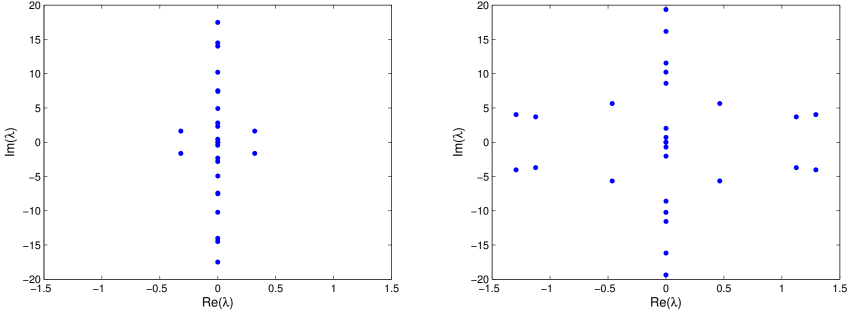


Figure : Eigenvalues λ of the spectral stability problem on the complex plane for the standing wave solutions $(u^+_{n,\omega}, 0)$ with $n = 1$ (a) and $n = 2$ (b) corresponding to $\omega = -1$.

Numerical solutions for $p = 1$

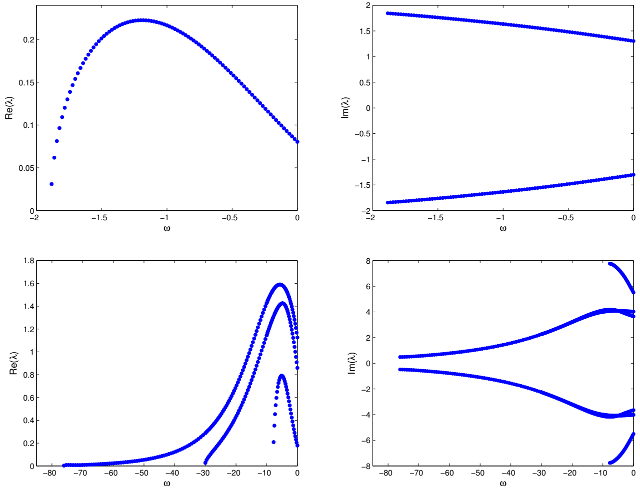


Figure : Real and imaginary parts of the unstable eigenvalues λ of the spectral stability problem versus parameter ω for the standing wave solutions $(u^{+}_{n,\omega}, 0)$ with $n = 1$ (a) and $n = 2$ (b).

Conclusions and Perspectives

- The classification of nonlinear bound states for the cubic NLS equation on a tadpole graph exhibits a variety of behaviors previously unknown for the standard NLS equation with power nonlinearity on the line.

- The analysis suggests soliton bifurcations from the edge of the continuum spectrum is a general feature when stationary states on a bounded interval are coupled with stationary states on the unbounded interval. Stability properties are accessible near the bifurcation.

- Complete the stability analysis:
  - Energy minimization properties for the primary branch
  - Spectral stability along the higher branches as $\omega \to -\infty$

- Other graphs: influence of geometry (or more complex topology)?