

Translationally invariant NLS lattices

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References:

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D.P., *Nonlinearity*, accepted (2006)

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Discrete nonlinear Schrödinger model

Continuous NLS model

$$iu_t = u_{xx} + |u|^2u, \quad x \in \mathbb{R}, \quad u \in \mathbb{C}$$

admits traveling pulse solutions

$$u(x, t) = \sqrt{\omega} \operatorname{sech}(\sqrt{\omega}(x - 2ct - s)) e^{ic(x-ct)+i\omega t+i\theta},$$

where $\omega \in \mathbb{R}_+$ and $(c, s, \theta) \in \mathbb{R}^3$.

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"Standard" (on-site) discretisation

$$i\dot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + |u_n|^2 u_n, \quad n \in \mathbb{Z}$$

does not have "true" traveling pulse solutions.

Reductions for traveling waves

Traveling waves

$$u_1(t) = u_0(t - \tau)e^{i\theta},$$

$$u_2(t) = u_1(t - \tau)e^{i\theta} = u_0(t - 2\tau)e^{2i\theta},$$

...

$$u_{n+1}(t) = u_n(t - \tau)e^{i\theta} = \dots = u_0(t - n\tau)e^{in\theta}$$

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Traveling solutions

$$u_n(t) = \phi(z)e^{i\omega t}, \quad z = hn - ct, \quad c = h/\tau, \quad \omega = c\theta/h.$$

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The differential advanced-delay equation

$$ic\phi'(z) = \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} - \omega\phi(z) + |\phi|^2\phi$$

Obstacles on existence

Classical solutions $\phi(z)$ on $z \in \mathbb{R}$

- $\phi(z)$ is $C^0(\mathbb{R})$ if $c = 0$
- $\phi(z)$ is $C^1(\mathbb{R})$ if $c \neq 0$
- $\phi(z)$ is a single-humped solution decaying to $\phi = 0$

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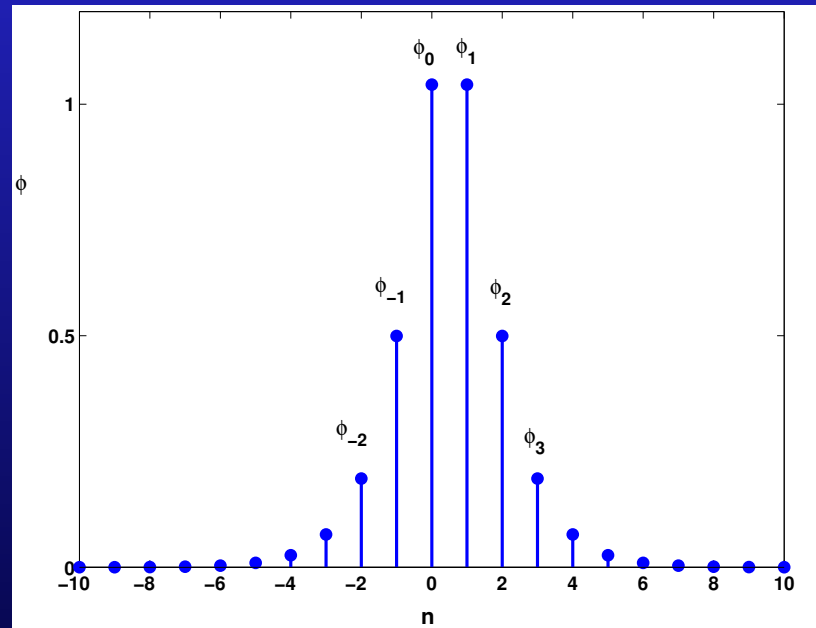
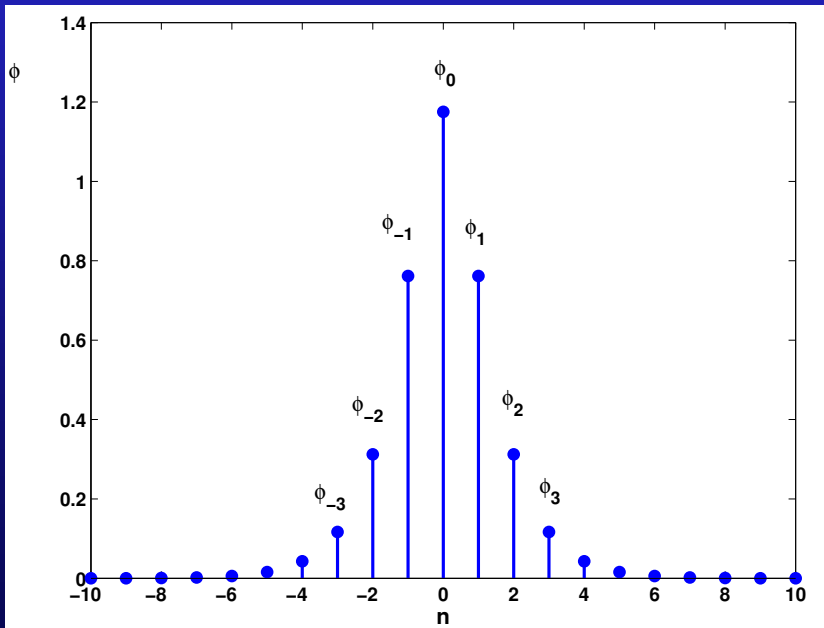
Properties of "standard" stationary solutions ($c = 0$):

- $\phi(z)$ is piecewise constant on $z \in \mathbb{R}$
- $\phi_n = \phi(nh)$ is symmetric either about a node or about the midpoint between two nodes
- No continuous deformation exists between these two particular solutions (Peierls–Nabarro potential)

Example of stationary solutions

Stationary solutions in the "standard" discrete NLS model

$$\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} - \phi_n + \phi_n^3 = 0, \quad n \in \mathbb{Z}$$



Exceptional discretizations

General discrete NLS equation:

$$i\dot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + f(u_{n-1}, u_n, u_{n+1})$$

where

P1 (continuity) $f(u, u, u) = 2|u|^2u$

P2 (symmetry) $f(v, u, w) = f(w, u, v)$

P3 (gauge) $f(e^{i\alpha}v, e^{i\alpha}u, e^{i\alpha}w) = e^{i\alpha}f(v, u, w) \forall \alpha \in \mathbb{R}$

P4 $f(v, u, w)$ is independent on h

P5 $f(v, u, w)$ is homogeneous cubic polynomial in (v, u, w)

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Exceptional nonlinearities are those that support continuous stationary solutions with $c = 0$ and $\phi \in C^0(\mathbb{R})$

Examples of exceptional discretizations

Ablowitz–Ladik lattice:

$$f = (u_{n+1} + u_{n-1}) |u_n|^2$$

New 2-parameter lattice:

$$f = (1 - \chi - 2\eta) |u_n|^2 (u_{n+1} + u_{n-1}) + \chi u_n^2 (\bar{u}_{n+1} + \bar{u}_{n-1}) \\ + \eta (|u_{n+1}|^2 + |u_{n-1}|^2) (u_{n+1} + u_{n-1})$$

Cases $(\chi, \eta) = (\frac{1}{2}, 0)$ and $(\chi, \eta) = (0, \frac{1}{2})$ are reported in
S. Dmitriev, P. Kevrekidis, A. Sukhorukov, et al.,
Phys. Lett. A 356, 324 (2006)

Purposes of this work

- Find the most general exceptional nonlinearity from the reduction of the second-order difference equation to the first-order difference equation.

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- Confirm that this reduction for *stationary* solutions is equivalent to conservation of momentum for *time-dependent* solutions (Kevrekidis, 2003), where the momentum is

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- Prove that this reduction gives a *sufficient* condition for existence of translationally invariant stationary solutions.
- Apply the normal form reduction (P, Rothos, 2005) as a *necessary* condition for existence of traveling solutions.

Reductions of difference equations

Consider the second-order difference equation

$$\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} - \omega\phi_n + f(\phi_{n-1}, \phi_n, \phi_{n+1}) = 0$$

and reduce the problem to the first-order difference equation

$$E_n = \frac{1}{h^2} |\phi_{n+1} - \phi_n|^2 - \frac{1}{2}\omega(\phi_n\bar{\phi}_{n+1} + \bar{\phi}_n\phi_{n+1}) + g(\phi_n, \phi_{n+1}) = E_0,$$

where

P1 (continuity) $g(u, u) = |u|^4$ P2 (symmetry) $g(u, w) = g(w, u)$

P3 (gauge) $g(e^{i\alpha}u, e^{i\alpha}w) = g(u, w) \quad \forall \alpha \in \mathbb{R}$

P4 $g(u, w)$ is independent on h

P5 $g(u, w)$ is homogeneous quartic polynomial in (u, w)

Constraints on the polynomial functions

The cubic polynomial f :

$$\begin{aligned} f &= \alpha_1 |u_n|^2 u_n + \alpha_2 |u_n|^2 (u_{n+1} + u_{n-1}) + \alpha_3 u_n^2 (\bar{u}_{n+1} + \bar{u}_{n-1}) \\ &+ \alpha_4 (|u_{n+1}|^2 + |u_{n-1}|^2) u_n + \alpha_5 (\bar{u}_{n+1} u_{n-1} + u_{n+1} \bar{u}_{n-1}) u_n \\ &+ \alpha_6 (u_{n+1}^2 + u_{n-1}^2) \bar{u}_n + \alpha_7 u_{n+1} u_{n-1} \bar{u}_n + \alpha_8 (|u_{n+1}|^2 u_{n+1} + |u_{n-1}|^2 u_{n-1}) \\ &+ \alpha_9 (u_{n+1}^2 \bar{u}_{n-1} + \bar{u}_{n+1} u_{n-1}^2) + \alpha_{10} (|u_{n+1}|^2 u_{n-1} + |u_{n-1}|^2 u_{n+1}), \end{aligned}$$

The quartic polynomial g :

$$\begin{aligned} g &= \gamma_1 (|\phi_n|^2 + |\phi_{n+1}|^2) (\bar{\phi}_{n+1} \phi_n + \phi_{n+1} \bar{\phi}_n) + \gamma_2 |\phi_n|^2 |\phi_{n+1}|^2 \\ &+ \gamma_3 (\phi_n^2 \bar{\phi}_{n+1}^2 + \bar{\phi}_n^2 \phi_{n+1}^2) + \gamma_4 (|\phi_n|^4 + |\phi_{n+1}|^4), \end{aligned}$$

The constraints for existence of reduction:

$$\alpha_4 = \alpha_1 - \alpha_6, \quad \alpha_5 = \alpha_6, \quad \alpha_7 = \alpha_1 - 2\alpha_6, \quad \alpha_{10} = \alpha_8 - \alpha_9$$

Remarks on conserved quantities

- These constraints are *equivalent* to the conditions for conservation of the momentum M :

$$M = i \sum_{n \in \mathbb{Z}} (\bar{u}_{n+1} u_n - u_{n+1} \bar{u}_n).$$

- These constraints are *incompatible* with the conditions for existence of the Hamiltonian structure:

$$i\dot{u}_n = \frac{\partial H}{\partial \bar{u}_n}, \quad H = \sum_{n \in \mathbb{Z}} \left(\frac{|u_{n+1} - u_n|^2}{h^2} - F(u_n, u_{n+1}) \right)$$

- These constraints *may provide* conservation of the power N

$$N = a \sum_{n \in \mathbb{Z}} |u_n|^2 + b \sum_{n \in \mathbb{Z}} (\bar{u}_{n+1} u_n + u_{n+1} \bar{u}_n)$$

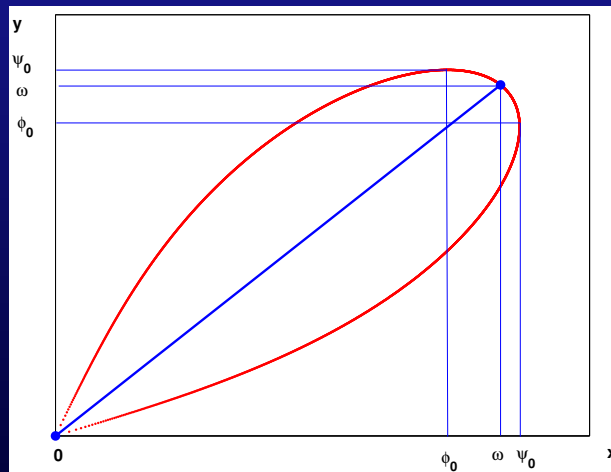
Continuous stationary solutions

Initial-value problem for real-valued solutions:

$$\begin{cases} (\phi_{n+1} - \phi_n)^2 = h^2 \omega \phi_n \phi_{n+1} - h^2 g(\phi_n, \phi_{n+1}), & n \in \mathbb{Z}, \\ \phi_0 = \varphi, \end{cases}$$

where

$$g(x, y) = \beta_1 x^2 y^2 + \beta_2 xy(x^2 + y^2) + \beta_3(x^4 + y^4)$$



Solutions of the first-order map

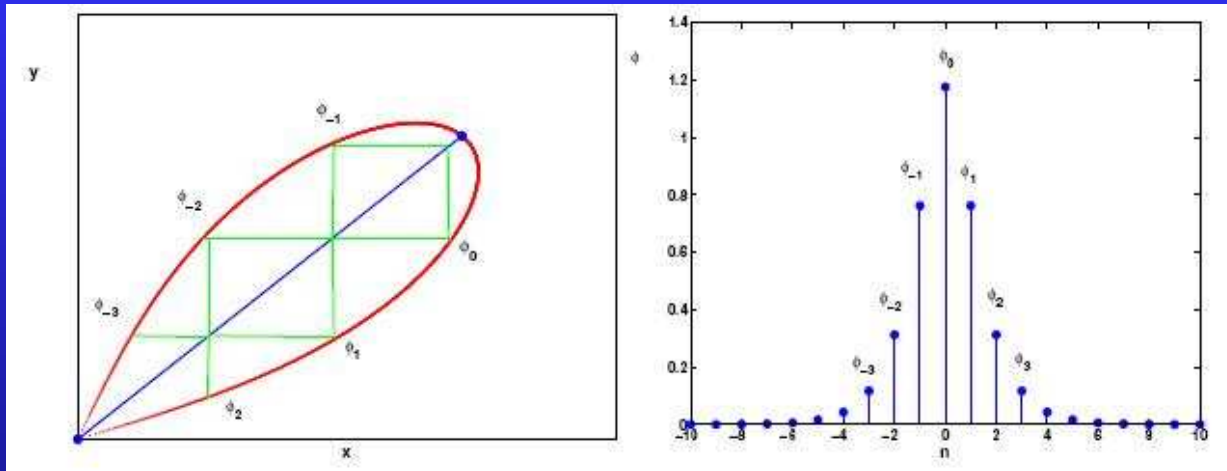
- There exists a unique monotonically decreasing sequence $\{\phi_n\}_{n=0}^{\infty}$ for any $0 < \phi_0 < \sqrt{\omega}$.
- There exists a unique monotonically increasing sequence $\{\phi_n\}_{n=-\infty}^0$ for any $0 < \phi_0 < \sqrt{\omega}$.

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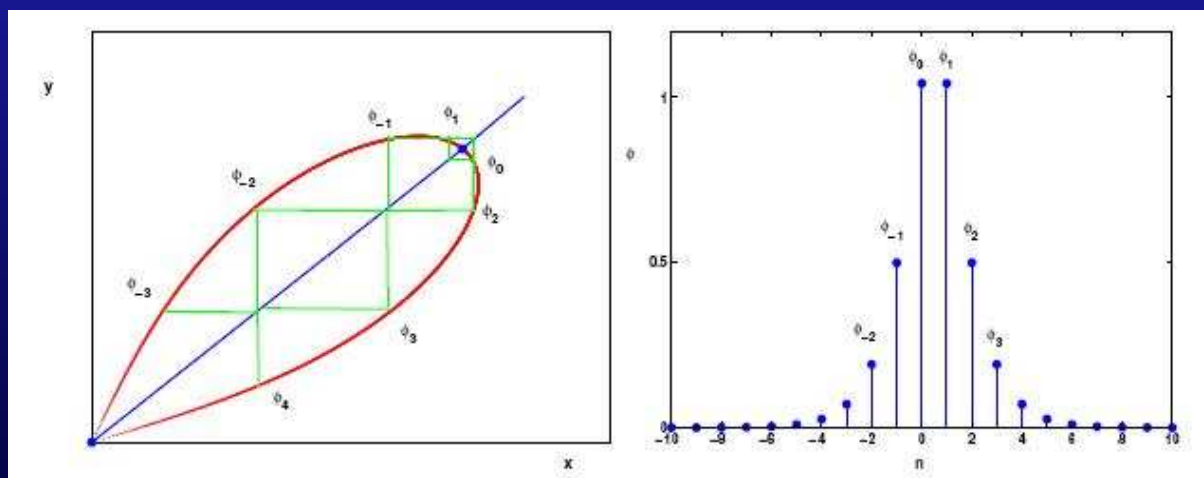
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- There exists a unique monotonically increasing sequence $\{\phi_n\}_{n=-\infty}^0$ for any $0 < \phi_0 < \sqrt{\omega}$.
- There exists a unique single-humped sequence $S_{\text{on}} = \{\phi_n\}_{n=\mathbb{Z}}$ for $\phi_0 = \phi_{\text{max}}$
- There exists a unique 2-site top single-humped sequence $S_{\text{off}} = \{\phi_n\}_{n=\mathbb{Z}}$ for $\phi_0 = \sqrt{\omega}$
- For any $\phi_0 \in (0, \phi_{\text{max}}) \setminus \{S_{\text{on}}, S_{\text{off}}\}$, there exists a unique non-symmetric single-humped sequence $\{\phi_n\}_{n=\mathbb{Z}}$ with $\phi_k \neq \phi_m$ for all $k \neq m$.

Solutions of the first-order map

S_{on} :



S_{off} :



Traveling solutions

The reduction to the first-order map gives a *sufficient* condition for existence of the translationally invariant stationary solutions and a *necessary* condition for existence of traveling solutions near $c = 0$.

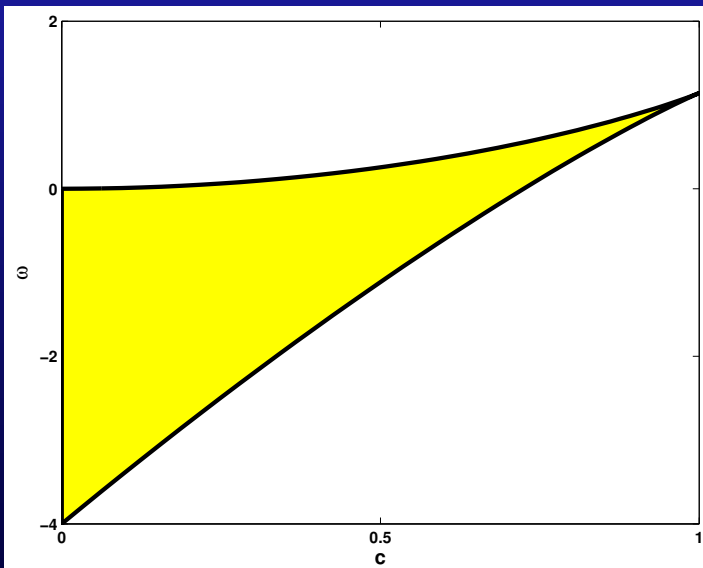
In other words, there exists $\phi(z) \in C^0(\mathbb{R})$ such that $\phi_n = \phi(hn - s)$ for $n \in \mathbb{Z}$ and $s \in \mathbb{R}$.

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Another *necessary* condition for existence of traveling solutions is derived (P, Rothos, 2005) near the particular point:



$$\omega = \frac{\pi - 2}{h^2}, \quad c = \frac{1}{h}$$

Reduction to the third-order ODE

Consider a transformation:

$$\phi(z) = \frac{\epsilon}{h} \Phi(\zeta) e^{\frac{i\pi z}{2h}}, \quad \zeta = \frac{\epsilon z}{h}, \quad c = \frac{1 + \epsilon^2 V}{h}, \quad \omega = \frac{\pi - 2 + \epsilon^2 \pi V + \epsilon^3 \Omega}{h^2}$$

which results in the differential advance-delay equation:

$$i(\Phi(\zeta + \epsilon) - \Phi(\zeta - \epsilon) - 2\epsilon\Phi'(\zeta)) = \epsilon^3 (2iV\Phi'(\zeta) + \Omega\Phi(\zeta)) - \epsilon^2 f(\dots)$$

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Apply Taylor series expansions:

$$\Phi(\zeta + \epsilon) - \Phi(\zeta - \epsilon) - 2\epsilon\Phi'(\zeta) = \frac{\epsilon^3}{3}\Phi'''(\zeta) + O(\epsilon^5),$$

$$f(\dots) = (\alpha_1 + 2\alpha_4 - 2\alpha_5 - 2\alpha_6 + \alpha_7) |\Phi|^2 \Phi + O(\epsilon).$$

Reduction to the third-order ODE

Since no single-humped localized solutions exist in

$$\frac{i}{3}\Phi''' - 2iV\Phi' - \Omega\Phi = |\Phi|^2\Phi,$$

the necessary condition for existence of traveling solutions is

$$\alpha_1 + 2\alpha_4 - 2\alpha_5 - 2\alpha_6 + \alpha_7 = 0.$$

The truncated third-order ODE is

$$\frac{i}{3}\Phi''' - 2iV\Phi' - \Omega\Phi + 2i|\Phi|^2\Phi' + i\gamma\Phi(|\Phi|^2)' = 0,$$

where γ is parameter.

Translationally invariant dNLS models

Parametrization of the dNLS model which gives translationally invariant solutions at $c = 0$ and $c = 1/h$:

$$\alpha_1 = 2\alpha_6, \alpha_4 = \alpha_5 = \alpha_6, \alpha_7 = 0, \alpha_{10} = \alpha_8 - \alpha_9,$$

subject to the normalization constraint:

$$\alpha_2 + \alpha_3 + 4\alpha_6 + 2\alpha_8 = 1.$$

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Additional conserved quantities:

- Conservation of power N gives four *one-parameter* models
- Conservation of density flux gives a *two-parameter* model

Open questions

Traveling solutions of the third-order ODE:

- $\gamma = 0$ - Hirota equation with 2-parameter solutions
- $\gamma = 1$ - Sasa-Satsuma equation with 2-parameter solutions
- $\gamma > -1$ - exact 1-parameter solutions (embedded solitons)

Can we prove *persistence* of any of these solutions in the full differential advance-delay equation?

Numerical approximation of traveling solutions is a work in progress.