

On Transverse Stability of Discrete Line Solitons

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Joint work with **Jianke Yang** (University of Vermont, USA)

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- ▶ In many Hamiltonian PDEs, one-dimensional solitons are unstable with respect to transverse perturbations:
 - ▶ Two-dimensional nonlinear Schrödinger equation

$$iu_t + u_{xx} \pm u_{yy} + |u|^2 u = 0.$$

- ▶ Dark solitons and KP-I equation

$$(u_t + uu_x + u_{xxx})_x = u_{yy}$$

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- ▶ Old works: Kadomtsev–Petviashvili (1970), Zakharov–Rubenchik (1971), Zakharov (1975), Pelinovsky–Stepanyants (1993), Bridges (2000).
- ▶ Recent works: Rousset–Tzvetkov (2008), Johnson–Zumbrun (2010), Stefanov–Stanislavova (2011), Haragus (2012), ...

Mathematical techniques

- ▶ Direct perturbation theory for eigenvalues
- ▶ Multi-symplectic geometric perturbation theory
- ▶ Evans function and algebraic perturbation theory
- ▶ Functional analysis framework and negative index theory (*)

Lattice NLS equation

The **discrete NLS (dNLS) equation**

$$i\dot{u}_{m,n} + \epsilon(u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n}) + |u_{m,n}|^2 u_{m,n} = 0,$$

where $(m, n) \in \mathbb{Z}^2$, $u_{m,n} \in \mathbb{C}$, and $\epsilon \in \mathbb{R}$.

The **Gross–Pitaevskii equation** with a periodic potential:

$$iu_t + u_{xx} + u_{yy} - V_0 \sin^2(x) \sin^2(y)u + |u|^2 u = 0,$$

where $(x, y) \in \mathbb{R}^2$, $u \in \mathbb{C}$, and $V_0 \in \mathbb{R}$.

Yang [PRA **84**, 033840 (2011)] found that line solitons can become stable with respect to transverse perturbations.

One-dimensional (stripe) dNLS lattice

$$i \frac{\partial u_m}{\partial t} + \epsilon(u_{m+1} + u_{m-1} - 2u_m) + \kappa \frac{\partial^2 u_m}{\partial y^2} + |u_m|^2 u_m = 0,$$

where $m \in \mathbb{Z}$, $y \in \mathbb{R}$, $u_m \in \mathbb{C}$, and $\epsilon, \kappa \in \mathbb{R}$.

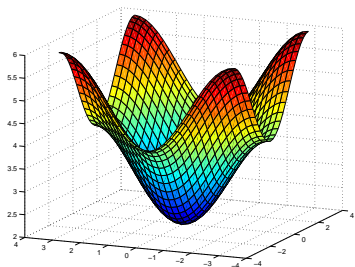
Yang *et al.* [Opt. Lett. **37**, 1571 (2012)] found again numerically that line solitons can become transversely stable.

Our objective is to study this phenomenon analytically by using the negative index theory.

Linearized dNLS equation:

$$i\dot{u}_{m,n} + \epsilon(u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{m,n}) = 0.$$

Bifurcations of stationary solitons occur from critical points of the dispersion surface, where $\nabla\omega = 0$.



Linear waves $e^{ikm+ipn-i\omega t}$ with $(k, p) \in [-\pi, \pi] \times [-\pi, \pi]$ satisfies the dispersion relation

$$\omega(k, p) = \epsilon(4 - 2\cos(k) - 2\cos(p))$$

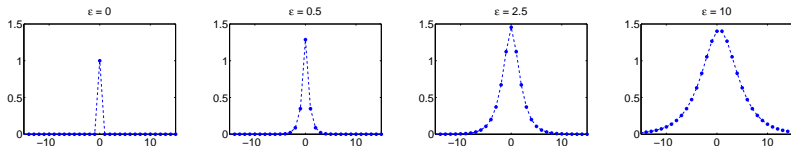
Critical points at $(0, 0)$, $(\pi, 0)$, $(0, \pi)$, and (π, π) .

Minimum point Γ : $k = p = 0, \omega(0, 0) = 0$

Line solitons $u_{m,n}(t) = e^{i\mu^2 t} \psi_m$ satisfy the 1D dNLS equation

$$-\mu^2 \psi_m + \epsilon(\psi_{m+1} + \psi_{m-1} - 2\psi_m) + |\psi_m|^2 \psi_m = 0,$$

A fundamental soliton exists for any $\epsilon > 0$ (Hermann, 2011)



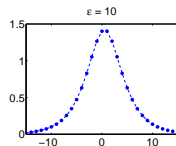
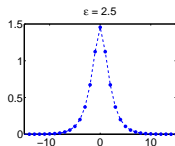
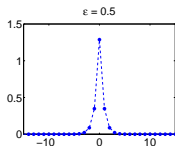
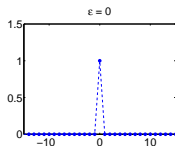
Continuous approximation $\psi_m \sim \sqrt{2} \mu \operatorname{sech} \left(\frac{\mu m}{\sqrt{\epsilon}} \right)$ as $\mu \rightarrow 0$
(Bambusi and Penati, 2010).

Saddle point $X : k = 0, p = \pi, \omega(0, \pi) = 4\epsilon$

Line solitons $u_{m,n}(t) = (-1)^n e^{i(\mu^2 - 4\epsilon)t} \psi_m$ satisfy the same 1D dNLS equation

$$-\mu^2 \psi_m + \epsilon(\psi_{m+1} + \psi_{m-1} - 2\psi_m) + |\psi_m|^2 \psi_m = 0,$$

Another family of line solitons exist.



Saddle point X' : $k = \pi$, $p = 0$, $\omega(\pi, 0) = 4\epsilon$

Line solitons $u_{m,n}(t) = (-1)^m e^{i(-\mu^2 - 4\epsilon)t} \psi_m$ satisfy the 1D dNLS equation

$$\mu^2 \psi_m - \epsilon(\psi_{m+1} + \psi_{m-1} - 2\psi_m) + |\psi_m|^2 \psi_m = 0.$$

No line solitons exist because

$$\mu^2 \|\psi\|_{l^2}^2 + \epsilon \langle \psi, (-\Delta)\psi \rangle + \|\psi\|_{l^4}^4 = 0$$

yields a contradiction.

Maximum point $M : k = \pi, p = \pi, \omega(\pi, \pi) = 8\epsilon$

Line solitons $u_{m,n}(t) = (-1)^{m+n} e^{i(-\mu^2 - 8\epsilon)t} \psi_m$ satisfy the same 1D dNLS equation

$$\mu^2 \psi_m - \epsilon(\psi_{m+1} + \psi_{m-1} - 2\psi_m) + |\psi_m|^2 \psi_m = 0.$$

No line solitons exist.

Minimum point Γ : $k = p = 0, \omega(0,0) = 0$

At the minimum point Γ , we can substitute

$$u_{m,n}(t) = U(X, Y, t)e^{i\mu^2 t}, \quad X = \frac{m}{\sqrt{\epsilon}}, \quad Y = \frac{n}{\sqrt{\epsilon}}$$

and obtain an elliptic 2D NLS equation as $\epsilon \rightarrow \infty$:

$$i\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} + (|U|^2 - \mu^2)U = 0.$$

Line solitons are unstable as $\epsilon \rightarrow \infty$.

Would the same be true for all $\epsilon > 0$?

Saddle point X : $k = 0$, $p = \pi$, $\omega(0, \pi) = 4\epsilon$

At the saddle point X , we can substitute

$$u_{m,n}(t) = (-1)^n U(X, Y, T) e^{i(\mu^2 - 4\epsilon)t}, \quad X = \frac{m}{\sqrt{\epsilon}}, \quad Y = \frac{n}{\sqrt{\epsilon}}$$

and obtain a hyperbolic 2D NLS equation as $\epsilon \rightarrow \infty$:

$$i \frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial X^2} - \frac{\partial^2 U}{\partial Y^2} + (|U|^2 - \mu^2)U = 0.$$

Line solitons are unstable as $\epsilon \rightarrow \infty$.

Would the same be true for all $\epsilon > 0$?

Instability Theorem

Linearizing at the discrete line soliton,

$$u_{m,n}(t) = e^{i\mu^2 t} [\psi_m + v_{m,n}(t)], \quad v_{m,n}(t) = e^{\lambda t + ipn} (U_m + iW_m),$$

we obtain the linear stability problem

$$L_+(p)U = -\lambda W, \quad L_-(p)W = \lambda U,$$

where

$$\begin{aligned} (L_+ U)_m &= -\epsilon [U_{m+1} + U_{m-1} + (2 \cos(p) - 4)U_m] + (\mu^2 - 3\psi_m^2)U_m, \\ (L_- W)_m &= -\epsilon [W_{m+1} + W_{m-1} + (2 \cos(p) - 4)W_m] + (\mu^2 - \psi_m^2)W_m. \end{aligned}$$

Fix $\mu = 1$ and consider a fundamental (positive, 1-humped) soliton:

$$\psi_m = \delta_{m,0} + \epsilon(\delta_{m,1} + \delta_{m,-1}) + \mathcal{O}(\epsilon^2).$$

Theorem

Consider the fundamental soliton bifurcating from the Γ point. For any $\epsilon > 0$, there is $p_0(\epsilon) \in (0, \pi]$ such that for any p with $0 < |p| < p_0(\epsilon)$ the linear-stability problem admits a pair of real eigenvalues $\pm\lambda(\epsilon, p)$ with $\lambda(\epsilon, p) > 0$.

In addition, $p_0(\epsilon) = \pi$ if $0 < \epsilon < \frac{1}{2}$. Furthermore, for any $p \in [-\pi, \pi]$, the eigenvalue $\lambda(\epsilon, p)$ has the following asymptotic expansion in the anti-continuum limit,

$$\lambda^2(\epsilon, p) = 8\epsilon \sin^2\left(\frac{p}{2}\right) + \mathcal{O}(\epsilon^2) \text{ as } \epsilon \rightarrow 0.$$

- ▶ We have

$$L_{\pm}(p) = L_{\pm}(0) + 2\epsilon [1 - \cos(p)] \geq L_{\pm}(0).$$

- ▶ $L_{-}(0)\psi = 0$ with $\psi > 0$. Hence $L_{-}(0) \geq 0$ and 0 is at the bottom of $L_{-}(0)$.
- ▶ By the perturbation theory, $L_{-}(p) > 0$ for all $p \neq 0$.
- ▶ $L_{+}(0)$ has at least one negative eigenvalue

$$\langle L_{+}(0)\psi, \psi \rangle = -2\|\psi\|_{l^4}^4 < 0,$$

moreover, there is only one negative eigenvalue for any $\epsilon > 0$.

- ▶ $L_{+}(p)$ has exactly one negative and no zero eigenvalues for small $p \neq 0$.

Negative Index Theory:

$$\begin{aligned} N_{\text{real}}^- + N_{\text{imag}}^- + N_{\text{comp}} &= n(L_+(p)) = 1, \\ N_{\text{real}}^+ + N_{\text{imag}}^- + N_{\text{comp}} &= n(L_-(p)) = 0, \quad p \neq 0, \end{aligned}$$

where

- ▶ N_{real}^+ (N_{real}^-) are the numbers of real positive eigenvalues λ with positive (negative) quadratic form $\langle L_+(p)U, U \rangle$ at the eigenvector (U, W) of the linear stability problem;
- ▶ N_{imag}^- is the number of purely imaginary eigenvalues λ with $\text{Im}(\lambda) > 0$ and negative quadratic form $\langle L_+(p)U, U \rangle$;
- ▶ N_{comp} is the number of complex eigenvalues λ with $\text{Re}(\lambda) > 0$ and $\text{Im}(\lambda) > 0$.

Hence

$$N_{\text{real}}^- = 1, \quad N_{\text{real}}^+ = N_{\text{imag}}^- = N_{\text{comp}} = 0.$$

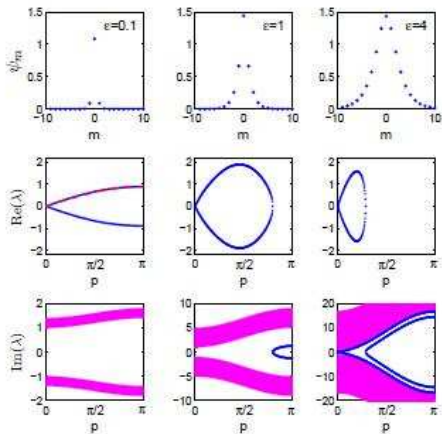


Figure : Left: $\epsilon = 0.1$; middle: $\epsilon = 1$; right: $\epsilon = 4$.

Stability Theorem

Linearizing at the discrete line soliton,

$$u_{m,n}(t) = (-1)^n e^{i(\mu^2 - 4\epsilon)t} [\psi_m + v_{m,n}(t)], \quad v_{m,n}(t) = e^{\lambda t + ipn} (U_m + iW_m)$$

we obtain the linear stability problem

$$L_+(p)U = -\lambda W, \quad L_-(p)W = \lambda U,$$

where

$$(L_+ U)_m = -\epsilon [U_{m+1} + U_{m-1} - 2 \cos(p) U_m] + (\mu^2 - 3\psi_m^2) U_m,$$
$$(L_- W)_m = -\epsilon [W_{m+1} + W_{m-1} - 2 \cos(p) W_m] + (\mu^2 - \psi_m^2) W_m.$$

Theorem

Consider the fundamental soliton bifurcating from the X point. There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ and $p \in [-\pi, \pi]$, the linear-stability problem does not admit any unstable eigenvalues but admits a pair of purely imaginary eigenvalues $\pm i\omega(\epsilon, p)$ of negative Krein signature.

For any $p \in [-\pi, \pi]$ and small ϵ , this eigenvalue $\omega(\epsilon, p)$ has the following asymptotic expression,

$$\omega^2(\epsilon, p) = 8\epsilon \sin^2\left(\frac{p}{2}\right) + \mathcal{O}(\epsilon^2) \text{ as } \epsilon \rightarrow 0.$$

- ▶ We have

$$L_{\pm}(p) = L_{\pm}(0) - 2\epsilon [1 - \cos(p)].$$

- ▶ $L_{-}(0)\psi = 0$ with $\psi > 0$. Hence $L_{-}(0) \geq 0$ and 0 is at the bottom of $L_{-}(0)$.
- ▶ By the perturbation theory, $L_{-}(p)$ has exactly one negative eigenvalue for small $\epsilon > 0$ and $p \neq 0$.
- ▶ $L_{+}(0)$ has exactly one negative eigenvalue and no zero eigenvalue for any $\epsilon > 0$.
- ▶ $L_{+}(p)$ has exactly one negative and no zero eigenvalues for small $\epsilon > 0$ and $p \neq 0$.

Negative Index Theory:

$$\begin{aligned} N_{\text{real}}^- + N_{\text{imag}}^- + N_{\text{comp}} &= n(L_+(p)) = 1, \\ N_{\text{real}}^+ + N_{\text{imag}}^- + N_{\text{comp}} &= n(L_-(p)) = 1, \end{aligned} \quad p \neq 0,$$

At $p = 0$, a double zero eigenvalue exists, which splits for $p \neq 0$ outside the continuous spectrum. Hence,

$$N_{\text{imag}}^- = 1, \quad N_{\text{real}}^+ = N_{\text{real}}^- = N_{\text{comp}} = 0,$$

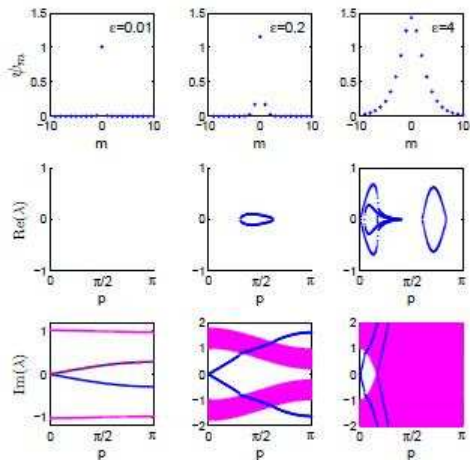


Figure : Left: $\epsilon = 0.01$; middle: $\epsilon = 0.2$; right: $\epsilon = 4$.

Consider the 1D Stripe dNLS lattice:

$$i \frac{\partial u_m}{\partial t} + \epsilon(u_{m+1} + u_{m-1} - 2u_m) + \kappa \frac{\partial^2 u_m}{\partial y^2} + |u_m|^2 u_m = 0, \quad m \in \mathbb{Z},$$

where $\epsilon > 0$ is small and $\kappa = \pm 1$.

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$$i \frac{\partial u_m}{\partial t} + \epsilon(u_{m+1} + u_{m-1} - 2u_m) + \kappa \frac{\partial^2 u_m}{\partial y^2} + |u_m|^2 u_m = 0, \quad m \in \mathbb{Z},$$

where $\epsilon > 0$ is small and $\kappa = \pm 1$.

Linearizing at the discrete line soliton,

$$u_m(y, t) = e^{i\mu^2 t} [\psi_m + v_m(y, t)], \quad v_m(y, t) = e^{\lambda t + ipy} (U_m + iW_m),$$

we obtain the linear stability problem

$$L_+(p)U = -\lambda W, \quad L_-(p)W = \lambda U,$$

where

$$\begin{aligned} (L_+(p)U)_m &= -\epsilon(U_{m+1} + U_{m-1} - 2U_m) + (\mu^2 + \kappa p^2 - 3\psi_m^2)U_m, \\ (L_-(p)W)_m &= -\epsilon(W_{m+1} + W_{m-1} - 2W_m) + (\mu^2 + \kappa p^2 - \psi_m^2)W_m. \end{aligned}$$

- ▶ At $\epsilon = 0$, the linear system has two semi-simple eigenvalue of infinite multiplicity at $\lambda = \pm i(1 + \kappa p^2)$ and two simple eigenvalues at $\lambda = \pm \sqrt{\kappa p^2(2 - \kappa p^2)}$.
- ▶ We also have

$$L_{\pm}(p) = L_{\pm}(0) + \kappa p^2.$$

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- ▶ We also have

$$L_{\pm}(p) = L_{\pm}(0) + \kappa p^2.$$

- ▶ For $\kappa = 1$ and $\epsilon = 0$, simple eigenvalues $\lambda = \pm p\sqrt{2 - p^2}$ are real for $p \in (0, \sqrt{2})$ and purely imaginary eigenvalues for $p > \sqrt{2}$ bounded away from the continuum spectrum.
- ▶ For small $\epsilon > 0$, the negative index count gives

$$N_{\text{real}}^- = 1, \quad p \in (0, p_0(\epsilon))$$

and

$$n(L_+(p)) = n(L_-(p)) = 0, \quad p > p_0(\epsilon),$$

where $p_0(\epsilon) = \sqrt{2} + \mathcal{O}(\epsilon)$.

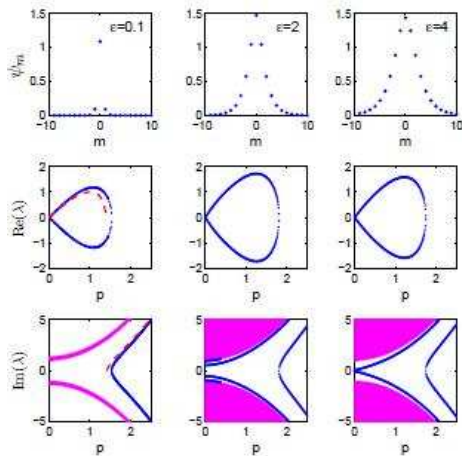


Figure : Left: $\epsilon = 0.1$; middle: $\epsilon = 2$; right: $\epsilon = 4$.

- ▶ For $\kappa = -1$ and $\epsilon = 0$, simple eigenvalues $\lambda = \pm ip\sqrt{2 + p^2}$ are in resonance with the essential spectrum $\lambda = \pm i(1 - p^2)$ at $p = p_c = \frac{1}{2}$.
- ▶ The simple eigenvalues have negative Krein signature and the essential spectrum has positive Krein signature for $p \in (-1, 1)$. For small $\epsilon > 0$, the resonance gives rise to complex instabilities with $N_{\text{comp}} = 1$ for p near p_c .
- ▶ Asymptotic theory gives

$$\lambda(\epsilon, p) = \frac{3}{4}i + \frac{i\epsilon}{15}(14 + 17\delta) + \frac{2\epsilon}{15}\sqrt{15 - 4(1 - 2\delta)^2} + \mathcal{O}(\epsilon^2),$$

where $\delta = (p^2 - p_c^2)/\epsilon$.

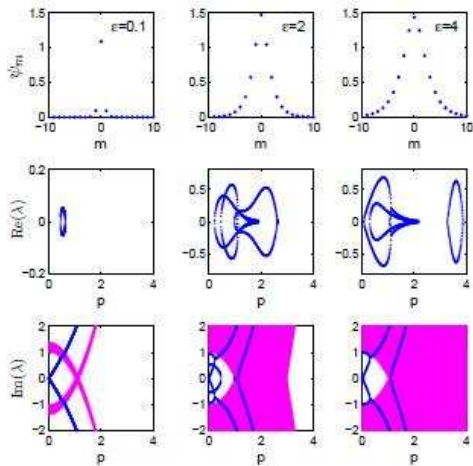


Figure : Left: $\epsilon = 0.1$; middle: $\epsilon = 2$; right: $\epsilon = 4$.

Summary

- ▶ Transverse stability problems are much easier than regular stability problems because symmetry-breaking perturbations remove kernels of the linearized operators.
- ▶ Applications of the negative index theory are developed in regular l^2 spaces, there is no necessity of constrained spaces.
- ▶ Lattice problems have additional simplifications near the anti-continuum limit, where asymptotic methods can be used in conjugation with the negative stability theory.
- ▶ Discretization may induce transverse stability of continuously unstable solitons. The role of discretization may be taken by the periodic potentials in the continuous NLS equations.