Transverse stability of periodic waves in KP-II

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Outline

The Kadomtsev–Petviashvili (KP) equation

It is a 2D generalization of the Korteweg-de Vries (KdV) equation:

$$(u_t + 6uu_x + u_{xxx})_x = \pm u_{yy}.$$

The plus/minus sign corresponds to KP-I/KP-II equations.
KP stands for B. Kadomtsev and V.I. Petviashvili, who derived this equation in 1970 to study transverse stability of 1D travelling waves.

Each sign is applicable as a model for fluid dynamics:

- KP-I for high surface tension (e.g., oil);
- KP-II for low surface tension (e.g., water).

1D periodic travelling waves

1D wave satisfies the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

Periodic travelling waves $u = \phi(x + ct)$ are found from the second-order ODE:

$$c\phi(x) + 3\phi(x)^2 + \phi''(x) = 0,$$

solutions are available in the cnoidal form with the Jacobian elliptic function *cn*.

KdV cnoidal waves are linearly and nonlinearly stable:

- N. Bottman, B. Deconinck, DCDS A (2009)
- B. Deconinck, T. Kapitula, Physics Letters A (2010)
- M. Nivala, B. Deconinck, Physica D (2010)

Transverse stability of periodic waves

Transverse stability of periodic waves is determined for small 2D perturbations w:

$$(w_t + cw_x + 6(\phi(x)w)_x + w_{xxx})_x = \pm w_{yy}.$$

or for $w(x,y,t) = W(x)e^{\lambda t + ipy}$ by the spectral problem

$$\lambda W_x + cW_{xx} + 6(\phi(x)W)_{xx} + W_{xxxx} \pm p^2 W = 0.$$

Functional-analytic results in the recent literature:

KP-I: Periodic and solitary waves are transversely unstable [Johnson–Zumbrun (2010); Rousset–Tzvetkov (2011); Hakkaev (2012)]

KP-II: Solitary waves are transversely stable [Mizumachi–Tzvetkov (2012); T. Mizumachi (2015) (2017)]

KP-II: Stability of periodic waves is open [M. Haragus (2010)].

Main result for KP-II

Rewrite the spectral problem as $A_{c,p}(\lambda)W = 0$, where

$$A_{c,p}(\lambda)W := \lambda W_x + cW_{xx} + 6(\phi(x)W)_{xx} + W_{xxxx} - p^2 W.$$

Theorem (M.Haragus–J.Li–D.P, 2017)

For every $p \neq 0$, the linear operator $A_{c,p}(\lambda)$ is invertible in $C_b(\mathbb{R})$ for any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$. Consequently, the periodic travelling wave is transversely spectrally stable with respect to 2D bounded perturbations.

Forgotten results on spectral transverse stability of periodic waves in KP-II:

- E.A. Kuznetsov, M.D. Spector, and G. E. Falkovich, Physica D (1984).
- M.D. Spector, Sov. Phys. JETP (1988).

Eigenfunctions of spectral problem are computed explicitly and completeness of eigenfunction is analyzed formally.

KP-II as an integrable evolution equation

KP-II

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0.$$

is integrable in the sense of the inverse scattering transform method

- The (smooth) solution $u(x, y, t)$ is a potential of the Lax operator pair

$$L(u)\psi = \psi_y - \psi_{xx} - u\psi = \lambda\psi, \quad \frac{\partial \psi}{\partial t} = A(u, \lambda)\psi,$$

  such that λ is (x, y, t)-independent. The Cauchy problem can be solved by a sequence of direct and inverse scattering transforms.

- Infinitely many conserved quantities exist for smooth solutions.

- Bäcklund–Darboux transformation (dressing method) allows to construct many exact solutions.

V.E.Zakharov–A.B.Shabat (1974), M.J.Ablowitz–A.S.Fokas (1984), $+\infty$.

Conserved quantities for KP-II equation

KP-II

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0.$$

is a Hamiltonian system with conserved momentum

$$Q(u) = \frac{1}{2} \int u^2 dxdy$$

and energy

$$E(u) = \frac{1}{2} \int \left[u_x^2 - 2u^3 - (\partial_x^{-1} u_y)^2 \right] dxdy.$$

In particular,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \frac{\delta E}{\delta u}, \quad \text{where} \quad \frac{\delta E}{\delta u} = -u_{xx} - 3u^2 - \partial_x^{-2} u_{yy}.$$

$E(u)$ is sign-indefinite near $u = 0$ $\Rightarrow$ the energy method does not work for global well-posedness of KP-II in energy space.

Transverse spectral stability for periodic perturbations

Let $\phi(x+2\pi) = \phi(x)$, $c > 1$ be the periodic wave of KdV. Then, it is a critical point of $E(u) - cQ(u)$. Consider the spectral problem

$$A_{c,p}(\lambda)W = \lambda W_x + cW_{xx} + 6(\phi(x)W)_{xx} + W_{xxxx} - p^2 W = 0,$$

for $p \neq 0$ and $\mathrm{Re}(\lambda) > 0$. If $W \in L^2_{\mathrm{per}}(0,2\pi)$ is a solution for $p \neq 0$, then $W \in \dot{L}^2_{\mathrm{per}}(0,2\pi)$, the zero-mean subspace of $L^2_{\mathrm{per}}(0,2\pi)$.

Recall that ∂_x^{-1} is a bounded operator from $\dot{L}^2_{\mathrm{per}}(0,2\pi)$ to $\dot{L}^2_{\mathrm{per}}(0,2\pi)$ and rewrite $A_{c,p}(\lambda)W = 0$ formally as

$$\lambda W = \partial_x L_{c,p} W, \quad L_{c,p} := -\partial_x^2 - c - 6\phi(x) + p^2 \partial_x^{-2}.$$

The operator $L_{c,p} : H^2_{\mathrm{per}}(0,2\pi) \to L^2(0,2\pi)$ is self-adjoint,
In fact, $L_{c,p}$ is the Hessian operator of $E(u) - cQ(u)$.

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Spectral problem for periodic perturbations

The spectral problem is defined in $\dot{L}^2_{\rm per}(0,2\pi)$,

$$\lambda W = \partial_x L_{c,p} W, \quad L_{c,p} := -\partial_x^2 - c - 6\phi(x) + p^2 \partial_x^{-2}.$$

hence, strictly speaking, we shall write $\Pi_0 L_{c,p} \Pi_0$, where $\Pi_0 : L^2_{\rm per}(0,2\pi) \to \dot{L}^2_{\rm per}(0,2\pi)$ is the orthogonal projection operator.

Theorem (J.Bronski–M.Johnson–T.Kapitula, 2011)

If $\sigma(\Pi_0 L_{c,p} \Pi_0) \geq 0$, then no $\lambda \in \mathbb{C}$ with ${\rm Re}\lambda > 0$ exists.

Let us check the case $c = 1$, when $\phi = 0$. The spectrum of $\Pi_0 L_{c=1,p} \Pi_0$ is

$$\sigma(\Pi_0 L_{c=1,p} \Pi_0) = \{n^2 - 1 - p^2 n^{-2}, \quad n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$, there is a sufficiently large $p \in \mathbb{R}$ such that $n^2 - 1 - p^2 n^{-2} < 0$. The theorem above can not be applied.

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Spectral stability in 1D: $p = 0$

Similar problems occur in 1D, for the KdV equation, when perturbations are extended on the infinite line. Consider the spectral problem

$$\lambda W = \partial_x L_{c,p=0} W, \quad L_{c,p=0} := -\partial_x^2 - c - 6\phi(x),$$

where the perturbation W is defined in $L^2(\mathbb{R})$.

By using the Floquet theory for operators with 2π-periodic coefficients, we consider the periodic spectral problem

$$\lambda \tilde{W} = (\partial_x + i\gamma) L_{c,p=0}(\gamma) \tilde{W}, \quad L_{c,p=0}(\gamma) := -(\partial_x + i\gamma)^2 - c - 6\phi(x),$$

where the perturbation $\tilde{W}$ is now defined in $L^2_{\rm per}(0,2\pi)$ and $\gamma \in [0,1)$. Then, $\sigma(\partial_x L_{c,p=0})$ in $L^2(\mathbb{R})$ is the union of $\{\sigma((\partial_x + i\gamma) L_{c,p=0}(\gamma))\}_{\gamma \in [0,1)}$.

For $c = 1$, $\phi = 0$,

$$\sigma(L_{c=1,p=0}(\gamma)) = \{(n+\gamma)^2 - 1, \quad n \in \mathbb{N}\}, \quad \gamma \in (0,1).$$

The bands with $n = -1$ and $n = 0$ are negative. The same problem.

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An approach to prove orbital stability for KdV in 1D

Consider the higher-order energy

$$R(u) = \int \left[u_{xx}^2 - 10 u u_x^2 + 5 u^4 \right] dx.$$

which is constant for solutions of the KdV in H^2. The periodic wave ϕ is also a critical point of $R(u) - c^2 Q(u)$ and the associated Hessian operator

$$M_{c,p=0} = \partial_x^4 + 10 \partial_x \phi(x) \partial_x - 10 c \phi(x) - c^2.$$

$M_{c,p=0}$ is not positive either. However,...

Proposition (B.Deconinck–T.Kapitula, 2010)

For every $c > 1$, the operator $M_{c,p=0} - b L_{c,p=0}$ is positive for every $b \in (b_-(c), b_+(c))$, where

$$b_-(c) = \left[\frac{5}{3} + \frac{1 - 2k^2}{3\sqrt{1 - k^2 + k^4}} \right] c, \quad b_+(c) = \left[\frac{5}{3} + \frac{1 + k^2}{3\sqrt{1 - k^2 + k^4}} \right] c,$$

where $k \in (0,1)$ is the elliptic modulus for the cnoidal periodic waves.

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where $k \in (0,1)$ is the elliptic modulus for the cnoidal periodic waves.

A simple perturbative argument

For $c = 1$ and $\phi = 0$, we have

$$\begin{aligned} L_{c=1,p=0} &= -\partial_x^2 - 1, \\ M_{c=1,p=0} &= \partial_x^4 - 1. \end{aligned}$$

Therefore, the linear combination of the two Hessian operators

$$M_{c,p=0} - bL_{c,p=0} = \partial_x^4 + b\partial_x^2 + b - 1 = \left(\partial_x^2 + \frac{b}{2}\right)^2 - \left(1 - \frac{b}{2}\right)^2$$

is positive if $b = 2$. By perturbative computations, one can find a nonempty interval $(b_-(c), b_+(c))$ near $b = 2$ for $c > 1$.

From positivity of the combined Hessian operator and energy conservation of

$$\Lambda_b(u) := [R(u) - c^2 Q(u)] - b[E(u) - cQ(u)], \quad \text{e.g. } b = 2c,$$

orbital stability of 1D periodic waves in the KdV holds in Sobolev space $H^2_{\rm per}$ for any subharmonic periodic perturbation.

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Higher-order energy for KP-II equation

Recall the momentum and energy for KP-II:

$$Q(u) = \int u^2 dxdy, \quad E(u) = \int \left[u_x^2 - 2u^3 - (\partial_x^{-1} u_y)^2 \right] dxdy.$$

Periodic wave ϕ is a critical point of $E(u) - cQ(u)$.

Proposition (L.Molinet–J-C.Saut–N.Tzvetkov, 2007)

KP-II conserves the higher-order energy in H^2:

$$R(u) = \int \left[u_{xx}^2 - 10uu_x^2 + 5u^4 - \frac{10}{3} u_y^2 + \frac{5}{9} (\partial_x^{-2} u_{yy})^2 + \frac{10}{3} u^2 \partial_x^{-2} u_{yy} + \ldots \right] dxdy.$$

Periodic wave ϕ is a critical point of $R(u) - c^2 Q(u)$. However, no b exists so that ϕ is a minimum of $[R(u) - c^2 Q(u)] - b[E(u) - cQ(u)]$.

New approach - commuting linear operators

Recall the spectral problem in $\dot{L}^2_{\rm per}(0,2\pi)$:

$$\lambda W = \partial_x L_{c,p} W, \quad L_{c,p} := -\partial_x^2 - c - 6\phi(x) + p^2 \partial_x^{-2}.$$

Let us search for a self-adjoint operator $M_{c,p}$ in $\dot{L}^2_{\rm per}(0,2\pi)$ such that

$$L_{c,p}\partial_x M_{c,p} = M_{c,p}\partial_x L_{c,p}.$$

Theorem (M.Haragus–J.Li–D.P, 2017)

Assume that $M_{c,p} \geq 0$ and the kernel of $M_{c,p}$ is contained in the kernel of $L_{c,p}$. The spectrum of $\partial_x L_{c,p}$ in $\dot{L}^2_{\rm per}(0,2\pi)$ is purely imaginary.

An elementary proof

Theorem (M.Haragus–J.Li–D.P, 2017)

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Let $\lambda_0 \in \mathbb{C}$ with ${\rm Re}\lambda_0 \neq 0$ be a simple eigenvalue of the spectral problem:

$$\lambda_0 W_0 = \partial_x L_{c,p} W_0, \quad W_0 \in {\rm D}(\partial_x L_{c,p}) \subset \dot{L}^2_{\rm per}(0,2\pi).$$

Assume that $W_0 \in {\rm D}(L_{c,p}\partial_x M_{c,p})$ and $L_{c,p}\partial_x M_{c,p} = M_{c,p}\partial_x L_{c,p}$. Then,

$$\begin{aligned} \lambda_0 \langle M_{c,p} W_0, W_0 \rangle_{L^2} &= \langle M_{c,p} W_0, \partial_x L_{c,p} W_0 \rangle_{L^2} = -\langle L_{c,p}\partial_x M_{c,p} W_0, W_0 \rangle_{L^2} \\ &= -\langle M_{c,p}\partial_x L_{c,p} W_0, W_0 \rangle_{L^2} = -\bar{\lambda}_0 \langle M_{c,p} W_0, W_0 \rangle_{L^2}, \end{aligned}$$

and since $\lambda_0 + \bar{\lambda}_0 \neq 0$, then $\langle M_{c,p} W_0, W_0 \rangle_{L^2} = 0$. Since $M_{c,p} \geq 0$, then $W_0 \in \ker(M_{c,p})$) but then $W_0 \in \ker(L_{c,p})$ so that $\lambda_0 = 0$.

Algorithmic search of the commuting operator

From the existence of the higher-order variational problem $R(u) - c^2 Q(u)$ associated with the higher-order energy of KP-II, we have one option for operator $M_{c,p}$:

$$M_{c,p} = \partial_x^4 + 10\partial_x \phi(x)\partial_x - 10c\phi(x) - c^2 - \frac{10}{3}p^2\left(1 + \phi(x)\partial_x^{-2} + \partial_x^{-1}\phi(x)\partial_x^{-1} + \partial_x^{-2}\phi(x)\right) + \frac{5}{9}p^4\partial_x^{-4}.$$

Then, $L_{c,p}\partial_x M_{c,p} = M_{c,p}\partial_x L_{c,p}$. However,

Proposition

For every $p \neq 0$, no value of $b \in \mathbb{R}$ exists such that $M_{c,p} - bL_{c,p}$ is positive in $L^2(\mathbb{R})$.

This outcome is related to bad (sign-indefinite) properties of $E(u)$ and $R(u)$ near $u = 0$.

Algorithmic search of the commuting operator

Let us search for another operator $M_{c,p}$ to satisfy the commutability relation

$$L_{c,p}\partial_x M_{c,p} = M_{c,p}\partial_x L_{c,p}.$$

By using symbolic computations, we have found

$$M_{c,p} = \partial_x^4 + 10\partial_x\phi(x)\partial_x - 10c\phi(x) - c^2 + \frac{5}{3}p^2\left(1 + c\partial_x^{-2}\right).$$

Then,

$$M_{c,p} - bL_{c,p} = M_{c,p=0} - bL_{c,p=0} + \frac{5}{3}p^2 - \left(b - \frac{5c}{3}\right)p^2\partial_x^{-2}.$$

Proposition

The operator $M_{c,p} - 2cL_{c,p}$ is positive in $L^2(\mathbb{R})$ for every $p \in \mathbb{R}$.

The periodic travelling wave v of the KP-II equation is spectrally stable with respect to two-dimensional bounded perturbations.

Conclusion

- Energy method does not work for KP-II.
- Spectral stability is obtained from commuting linear operators via symplectic structure.
- Linear orbital stability is obtained from coercivity of the quadratic form associated with the commuting linear operators.

$$\langle (M_c - 2cL_c)W, W\rangle_{L^2_{\rm per}} \geq C\|W\|^2_{L^2_{\rm per}}, \quad \langle W, \phi'\rangle_{L^2_{\rm per}} = 0.$$

for $W \in L^2_{\rm per}((0,2\pi N)\times(0,L))$ for every $N \in \mathbb{N}$ and every $L > 0$.

Open questions

- How is $M_{c,p}$ related to conserved quantities of the KP-II?
- Can we extend the proof to nonlinear orbital stability of periodic waves in the KP-II?
- Can we find commuting linear operators for non-integrable versions of nonlinear evolution equations?