

# Traveling waves in nonlinear lattices

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## References:

D.P., V. Rothos, *Physica D* 202, 16-36 (2005)

I. Barashenkov, O. Oxtoby, D.P., *Physical Review E* 72, 035602(R) (2005)

O. Oxtoby, D.P., I. Barashenkov, *Nonlinearity* 19, 217-235 (2006)

G. Iooss, D.P., *Physica D*, published online (2006)

D.P., *Nonlinearity*, submitted (2006)

Fields Institute, April 28-29, 2006

# Discrete $\phi^4$ model

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"Standard" (on-site) discretisation:

$$\ddot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + u_n(1 - u_n^2), \quad n \in \mathbb{Z}$$

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# Reductions for traveling waves

Traveling waves

$$u_1(t) = u_0(t - \tau),$$

$$u_2(t) = u_1(t - \tau) = u_0(t - 2\tau),$$

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The differential advanced-delay equation

$$c^2 \phi'' = \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} + \phi(1 - \phi^2)$$

# Obstacles on existence

Classical solutions  $\phi(z)$  on  $z \in \mathbb{R}$

- $\phi(z)$  is  $C^0(\mathbb{R})$  if  $c = 0$
- $\phi(z)$  is  $C^2(\mathbb{R})$  if  $c \neq 0$
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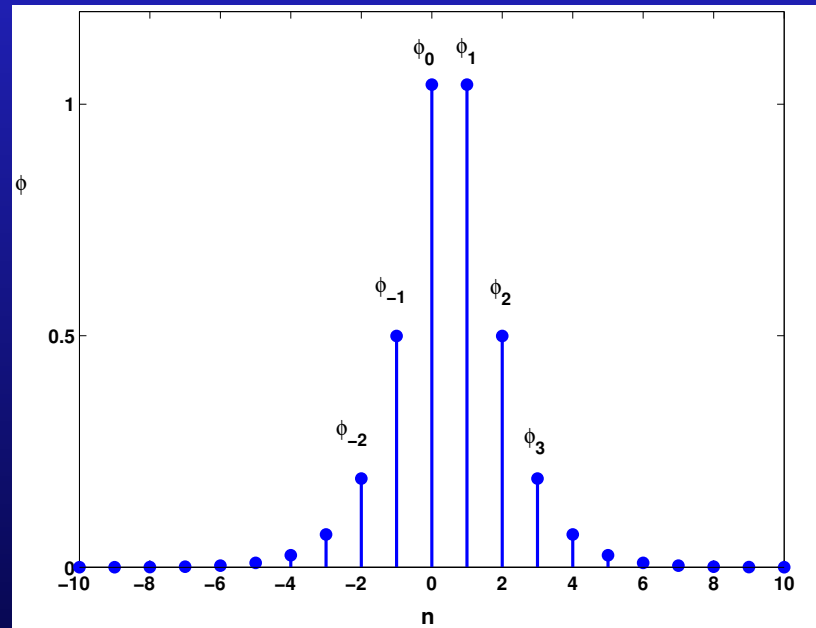
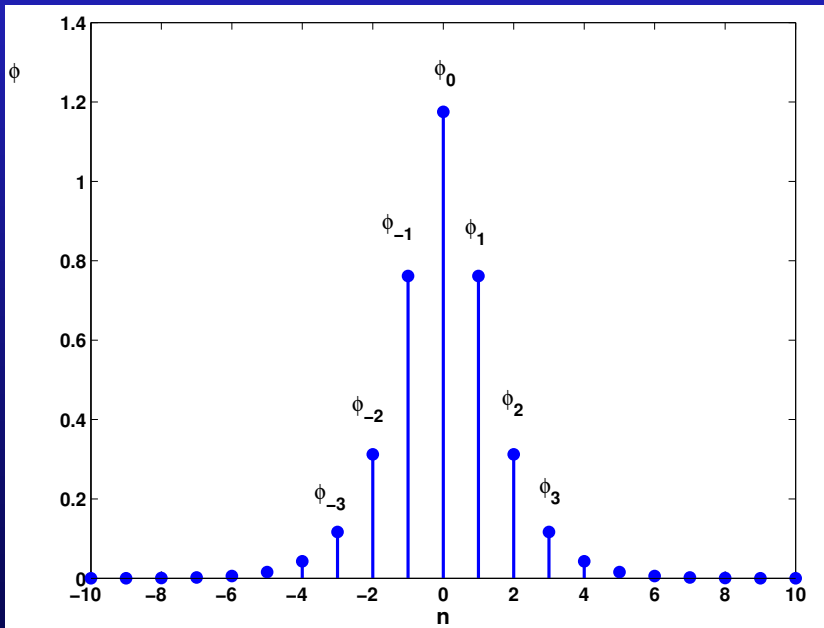
Stationary solutions ( $c = 0$ ) in standard discretizations:

- $\phi(z)$  is piecewise constant on  $z \in \mathbb{R}$
- $\phi_n = \phi(nh)$  is odd either about  $n = 0$  or about the midpoint between  $n = 0$  and  $n = 1$
- No continuous deformation exists between two particular solutions (Peierls–Nabarro potential)

# Example of stationary solutions

Stationary solutions in the standard discrete NLS model

$$\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} + \phi_n^3 - \phi_n = 0, \quad n \in \mathbb{Z}$$



# General and exceptional discretizations

General discrete  $\phi^4$  model:

$$\ddot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + f(u_{n-1}, u_n, u_{n+1})$$

where

P1 (continuity)  $f(u, u, u) = u(1 - u^2)$

P2 (symmetry)  $f(v, u, w) = f(w, u, v)$

P3  $f(v, u, w)$  is independent on  $h$

P4  $f(v, u, w) = u - Q(v, u, w)$ , where  $Q = O(3)$

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*Exceptional* nonlinearities are those that support continuous stationary solutions with  $c = 0$  and  $\phi \in C^0(\mathbb{R})$

# Examples of exceptional discretizations

Tovbis (1997)  $f = (u_{n+1} + u_{n-1}) (1 - u_n^2),$

Speight (1997)  $f = (2u_n + u_{n+1}) (3 - u_n^2 - u_n u_{n+1} - u_{n+1}^2) + \{n + 1 \rightarrow n\},$

Kevrekidis (2003)  $f = (u_{n+1} + u_{n-1}) (2 - u_{n+1}^2 - u_{n-1}^2).$



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Exceptional stationary solutions:

- The stationary solution has a translation parameter, e.g.

$$u_n = \tanh(a(hn - s)), \quad a = \frac{1}{h} \arcsin(h/2)$$

- Radiation from moving kinks is reduced

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- Q: Can we characterize all possible bifurcations of steadily moving kinks?
- A: YES, when the center manifold is finite-dimensional (when  $c \neq 0$ ).

# Families of exceptional discretizations

Consider the second-order difference map for stationary solutions

$$\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} + f(\phi_{n-1}, \phi_n, \phi_{n+1}) = 0$$

and reduce the problem to the first-order difference map

$$E = \frac{\phi_{n+1} - \phi_n}{h} - g(\phi_n, \phi_{n+1}) = \text{const}$$

Such discretizations with *polynomial* functions  $g(\phi_n, \phi_{n+1})$  exist for *exceptional* polynomial functions  $f(\phi_{n-1}, \phi_n, \phi_{n+1})$ .



# Continuous stationary solutions

**Theorem**[Speight, 1999]: Let  $g(\phi_n, \phi_{n+1})$  be a polynomial such that  $g(\phi, \phi) = \frac{1}{2}(1 - \phi^2)$ . Then, for any  $-1 < \phi_0 < 1$ , there exists a unique monotonic sequence  $\{\phi_n\}_{n \in \mathbb{Z}}$  such that

$$\phi_n < \phi_{n+1}, \quad \lim_{n \rightarrow \pm\infty} \phi_n = \pm 1$$

and  $\{\phi_n\}_{n \in \mathbb{Z}}$  is continuous in  $\phi_0$ .

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**Corollary:** There exists a  $C^0(\mathbb{R})$  monotonic kink solution  $\phi(z - s)$  of the advance-delay equation

$$\frac{\phi(z - h) - 2\phi(z) + \phi(z + h)}{h^2} + f(\phi(z - h), \phi(z), \phi(z + h)) = 0,$$

such that  $\lim_{z \rightarrow \pm\infty} \phi(z) = \pm 1$  and  $\phi_0 = \phi(-s)$ .

# Traveling solutions with $c \neq 0$ ?

Solutions  $\phi(z) = e^{\lambda z}$  of the linearized equation at  $\phi = 0$ :

$$c^2 \phi''(z) = \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} + \phi(z)$$

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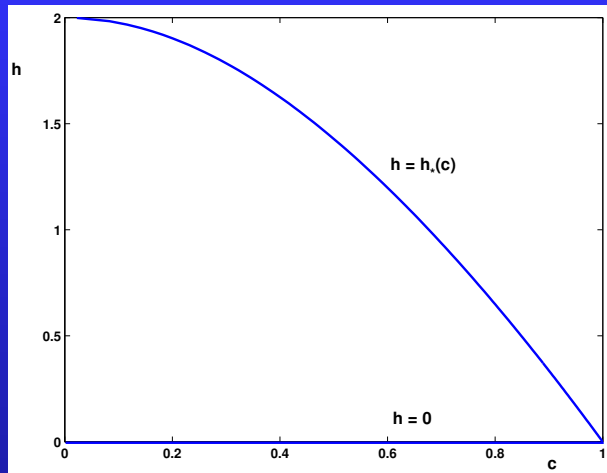
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Roots on the imaginary axis  $\Lambda = 2iK$ ,  $K \in \mathbb{R}$

$$\sin^2 K = \frac{h^2}{4} + c^2 K^2$$

# Parameter plane $(c, h)$ and bifurcations



Three bifurcations:

$$c = 0, 0 < h < 2$$

$$h = 0, 0 < c < 1$$

$$h = h_*(c), 0 < c < 1$$

- $c = 0, 0 < h < 2$ : All roots of  $K$  are real and simple.
- $h = 0, 0 < c < 1$ : Double zero coexists with finitely many pairs of real roots  $K$ .
- $h = h_*(c), 0 < c < 1$ : One pair of double real (non-zero) roots  $K$  exist (1:1 resonant Hopf bifurcation).

# Bifurcation at $h = 0$

Differential advance-delay equation:

$$c^2 \phi''(z) = \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} + \phi(z) - Q(\phi(z-h), \phi(z), \phi(z+h))$$

Formal perturbation expansion:

$$\phi(z+h) - 2\phi(z) + \phi(z-h) = h^2 \phi''(z) + \sum_{n=2}^{\infty} \frac{2}{(2n)!} \phi^{(2n)}(z) h^{2n}$$

and

$$\phi(z) = \sum_{n=0}^{\infty} h^{2n} \phi_{2n}(z)$$

# Beyond all order asymptotics

At the leading order  $\mathcal{O}(h^0)$ :

$$(1-c^2)\phi_0'' + \phi_0(1-\phi_0^2) = 0; \quad \Rightarrow \quad \phi_0(z) = \tanh \theta, \quad \theta = \frac{z}{2\sqrt{1-c^2}}.$$



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At the higher orders  $\mathcal{O}(h^{2n})$ :

$$\mathcal{L}\phi_{2n} = (\text{odd inhomogeneous terms})$$

where  $\mathcal{L} = -\frac{d^2}{d\theta^2} + 4 - 6 \operatorname{sech}^2 \theta$  with  $\mathcal{L}\operatorname{sech}^2 \theta = 0$ .

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To all orders, the perturbation expansion exists and

$$\phi(z) \rightarrow \pm 1 \quad \text{as} \quad |z| \rightarrow \infty.$$

Does it exist **beyond all orders** of the perturbation expansion?

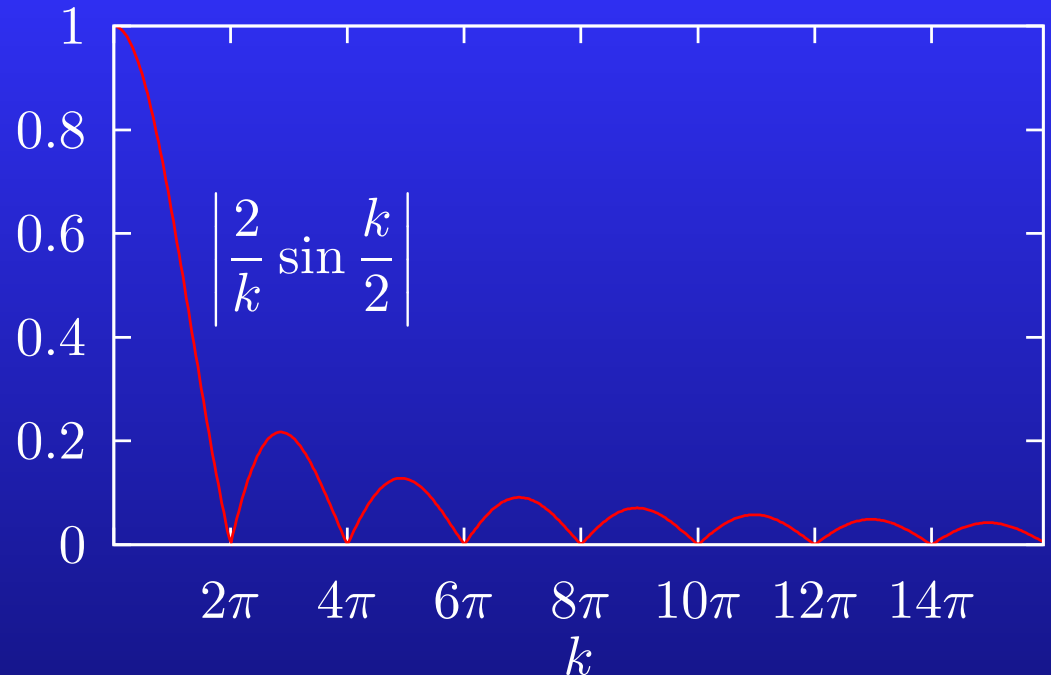
# Beyond all order asymptotics

Fourier oscillatory modes:

$$\phi(z) = \pm 1 + \epsilon e^{ikz/h}$$

has wavenumber  $k$  where

$$\sin^2 k = c^2 k^2 \text{ as } h \rightarrow 0.$$



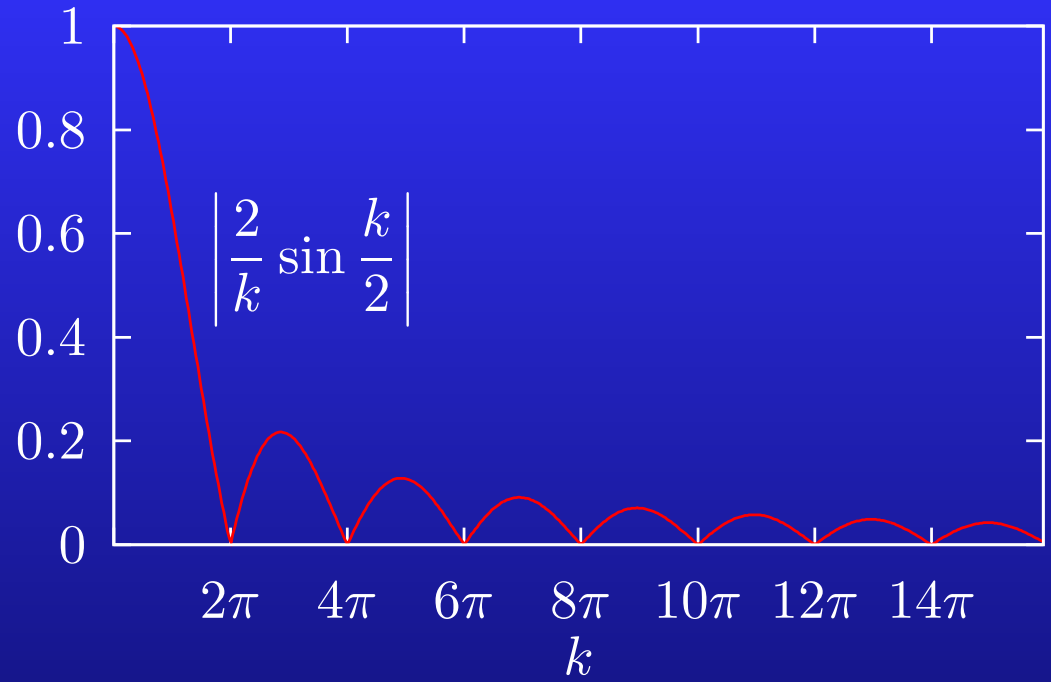
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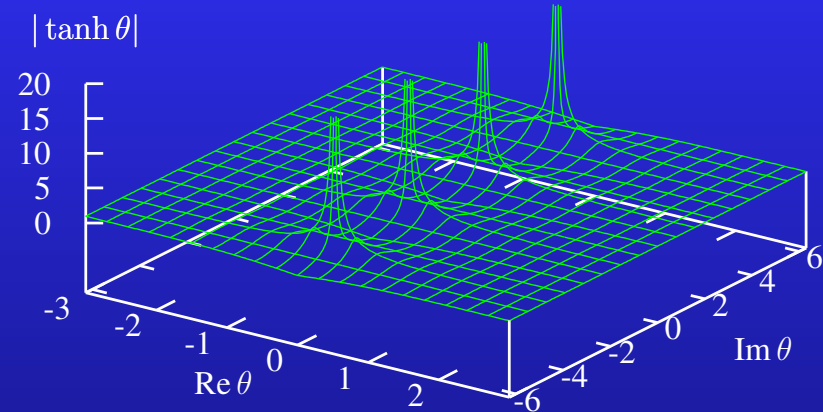
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Fourier modes do not occur in power series of  $h$ .

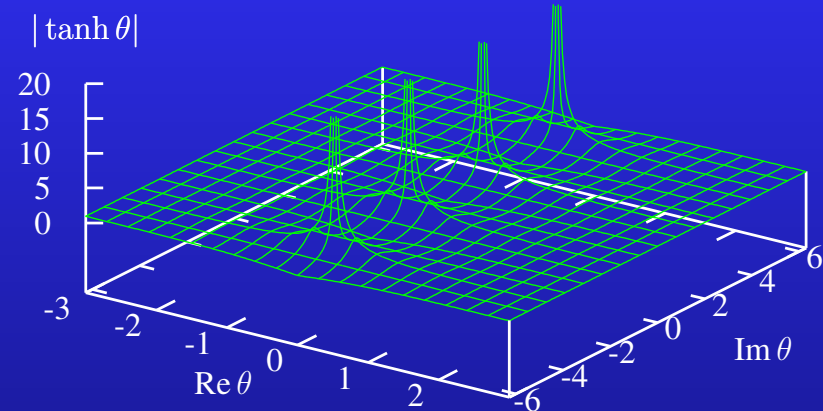
# Inner equation

Kruskal–Segur (1991):  
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Scaling transformation  $z = h\zeta + i\pi\sqrt{1 - c^2}$  and  $\phi(z) = \frac{1}{h}\psi(\zeta)$   
leads to the inner equation:

$$c^2\psi''(\zeta) = \psi(\zeta + 1) - 2\psi(\zeta) + \psi(\zeta - 1) - Q(\psi(\zeta - 1), \psi(\zeta), \psi(\zeta + 1)) + \frac{h^2}{2}\psi(\zeta).$$

# Inner asymptotic series

Let the solution  $\psi(\zeta)$  be expanded in powers of  $h^2$ :

$$\psi(\zeta) = \psi_0(\zeta) + \sum_{n=1}^{\infty} h^{2n} \psi_{2n}(\zeta).$$

Let the leading-order solution  $\psi_0(\zeta)$  be expanded in inverse power series of  $\zeta$ :

$$\psi_0(\zeta) = \sum_{n=0}^{\infty} \frac{a_{2n}}{\zeta^{2n+1}}, \quad a_0 = 2\sqrt{1-c^2}.$$

Theorem (Tovbis, 2000): If the inverse power series for  $\psi_0(\zeta)$  diverges, the formal perturbation expansion for  $\phi(z)$  diverges and some Fourier modes are non-zero beyond the formal expansion.

# Borel-Laplace transform

Let  $\psi_0(\zeta)$  be the Laplace transform of  $V_0(p)$ :

$$\psi_0(\zeta) = \int_{\gamma} V_0(p) e^{-p\zeta} dp$$

The resulting integral equation,

$$\left(4 \sinh^2 \frac{p}{2} - c^2 p^2\right) V_0(p) = \sum_{\alpha_1, \alpha_2, \alpha_3} a_{\alpha_1, \alpha_2, \alpha_3} e^{\alpha_1 p} V_0(p) * e^{\alpha_2 p} V_0(p) * e^{\alpha_3 p} V_0(p)$$

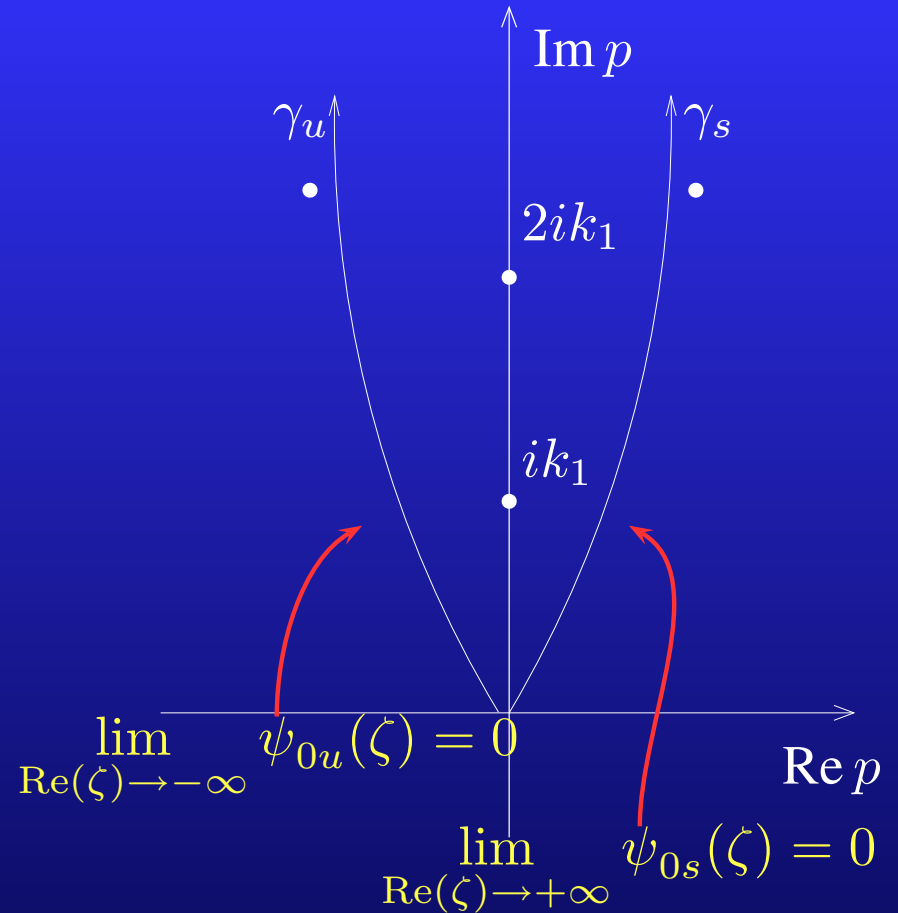
numerical coefficients

Power series solution:

$$V_0(p) = \sum_{n=0}^{\infty} v_{2n} p^{2n}, \quad v_0 = 2\sqrt{1 - c^2}.$$



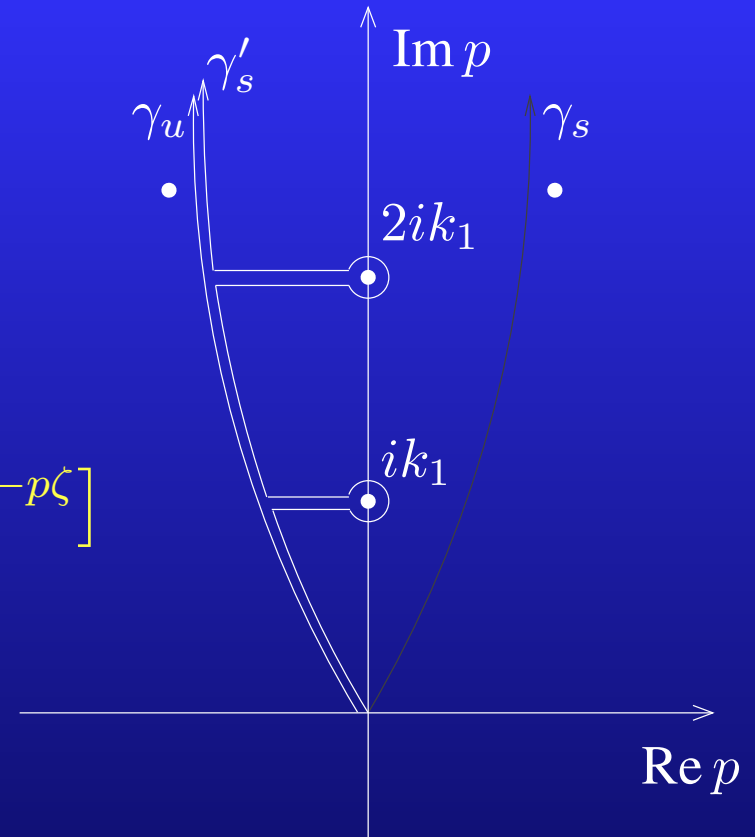
# Singularities of $V_0(p)$



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The distance between the stable and unstable manifold:

$$\psi_{0s}(\zeta) - \psi_{0u}(\zeta) = 2\pi i \sum \text{Res} [V_0(p)e^{-p\zeta}]$$



# The first pole of $V_0(p)$

Let  $p = ik_1$  be the nearest singularity to  $p = 0$ . Then,  $V_0(p)$  has the leading order behavior near  $p = ik_1$ :

$$V_0(p) \rightarrow \frac{k_1^2 K_1(c)}{(p^2 + k_1^2)},$$

where  $K_1(c)$  is referred to as the *Stokes constant*.

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Then

$$\psi_{0s}(\zeta) - \psi_{0u}(\zeta) = [\pi k_1 K_1(c) + \mathcal{O}(1/\zeta)] e^{-ik_1 \zeta}$$

Residue of the first pole  $p = ik_1$  can be deduced from a power-series solution for  $V_0(p)$  at  $p = 0$ , **which converges for  $|p| < k_1$ .**

# Numerical computations of $K_1(c)$

- Expand  $V_0(p)$  near the pole  $p \rightarrow \pm ik_1$ :

$$V_0(p) \rightarrow K_1(c) \sum_{n=0}^{\infty} (-1)^n k_1^{-2n} p^{2n}$$

- Expand  $V_0(p)$  in the power series at  $p = 0$ :

$$V_0(p) = \sum_{n=0}^{\infty} v_{2n} p^{2n}, \quad v_0 = 2\sqrt{1-c^2},$$

where the coefficients are obtained from a recurrence relation.

- Then  $K_1(c)$  can be obtained as a numerical limit:

$$K_1(c) = \lim_{n \rightarrow \infty} w_n, \quad w_n = (-1)^n k_1^{2n} v_{2n}$$

# Convergence of the algorithm

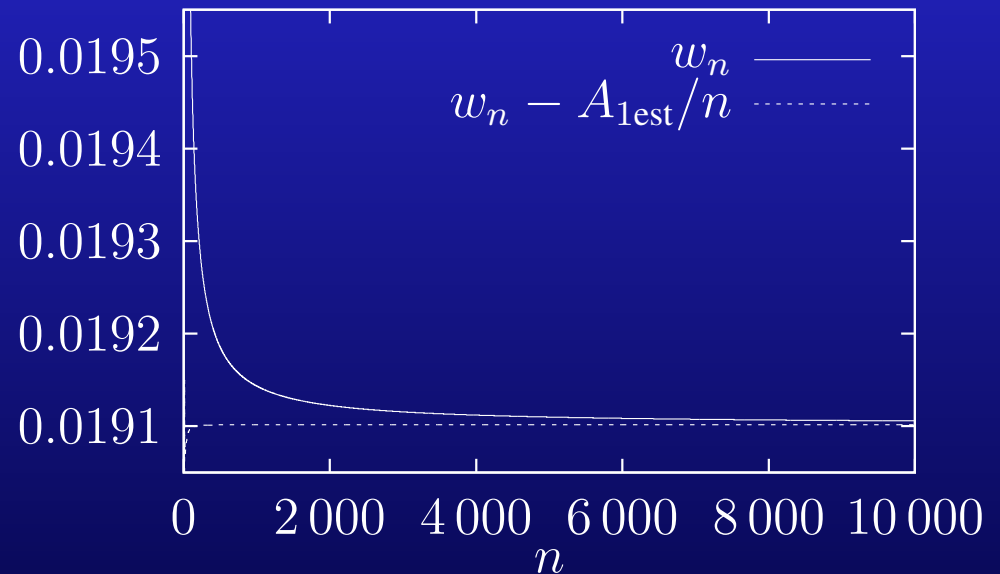
Convergence of  $w_n$  is slow:

$$w_n = K_1(c) + \frac{A(c)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

Kevrekidis' model with  $c = 0.5$ :

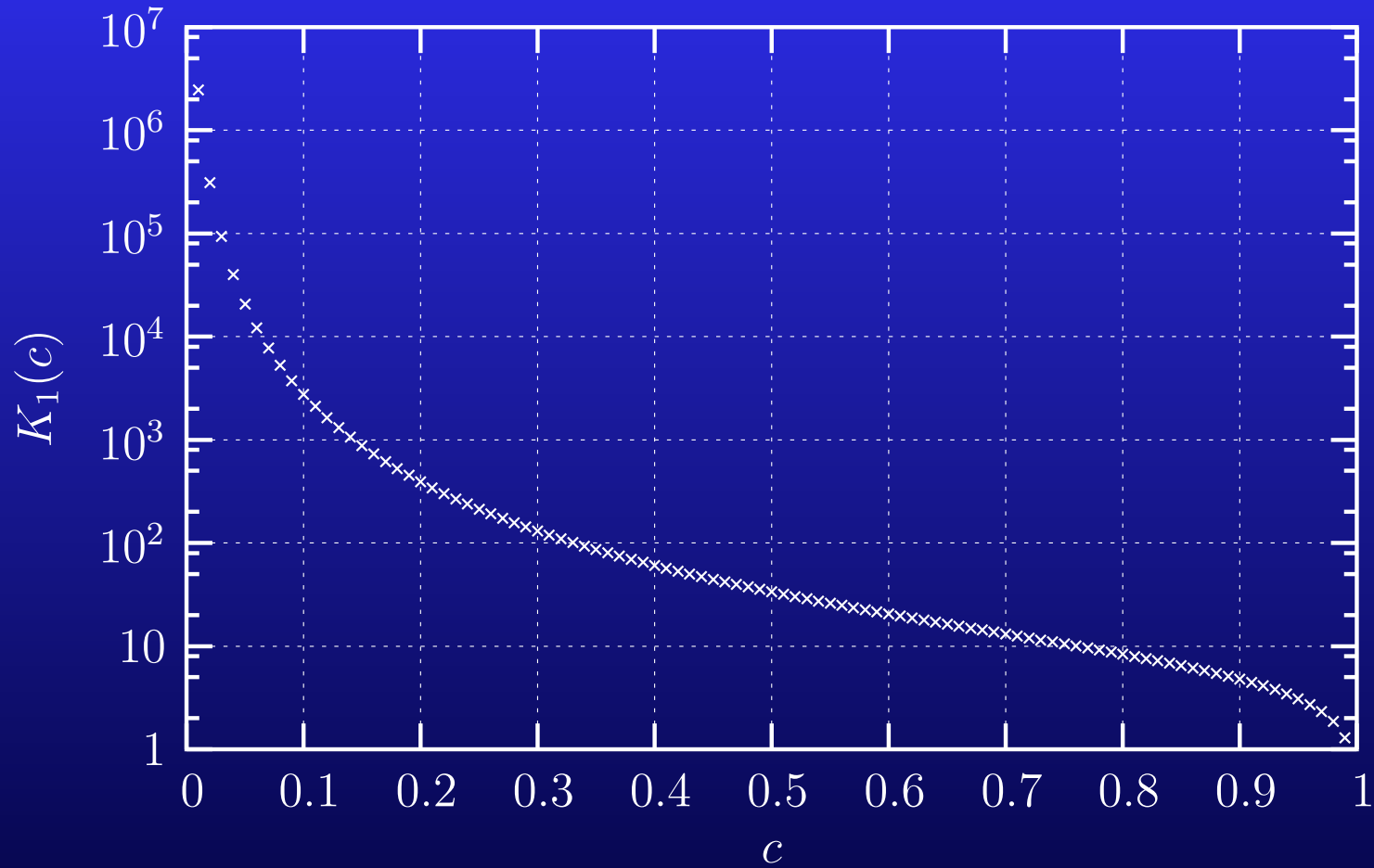
The error estimate of the numerical limit:

$$A_{\text{est}} = -n^2(w_n - w_{n-1})$$



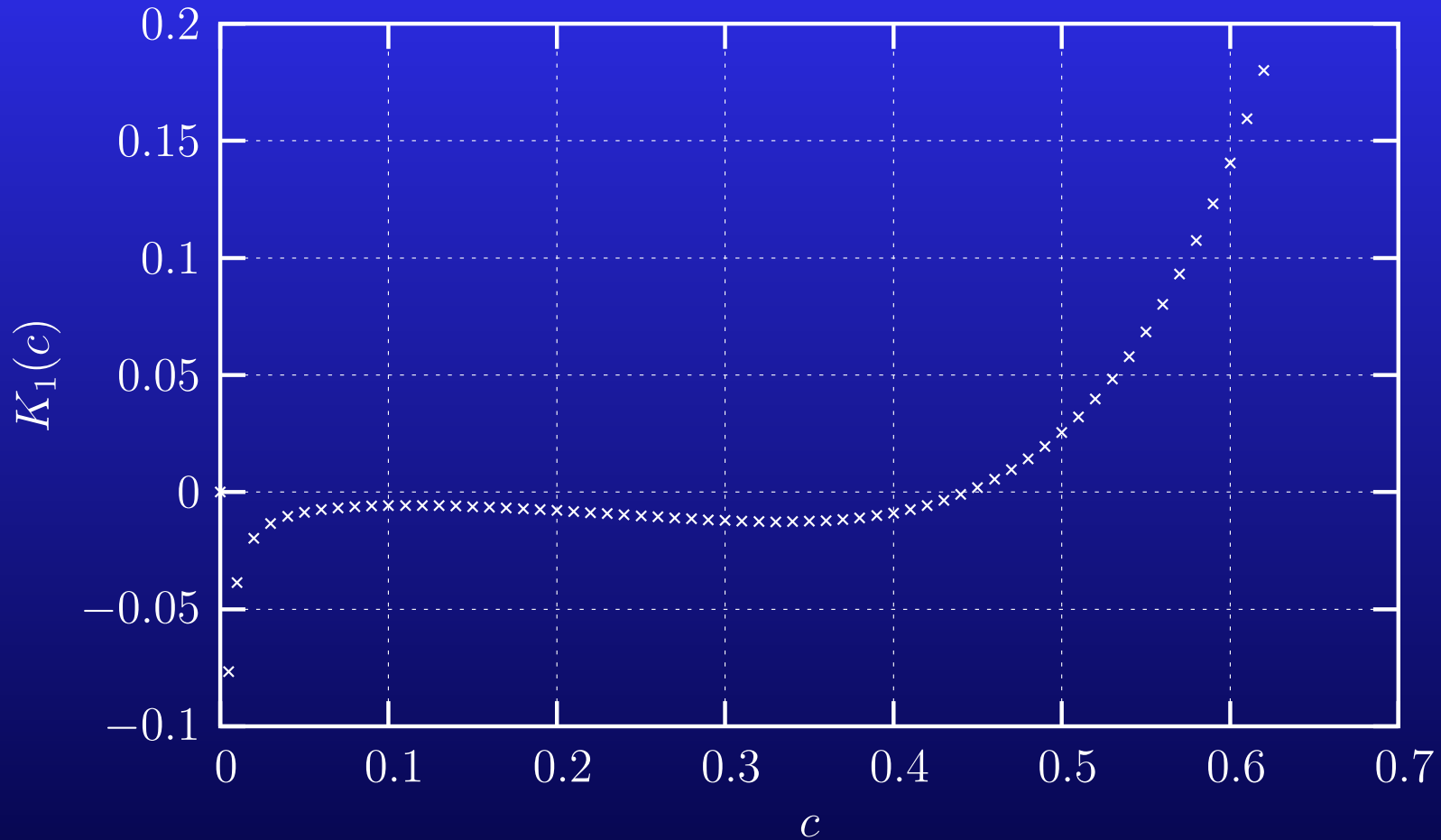
# Stokes Constant

Standard nonlinearity



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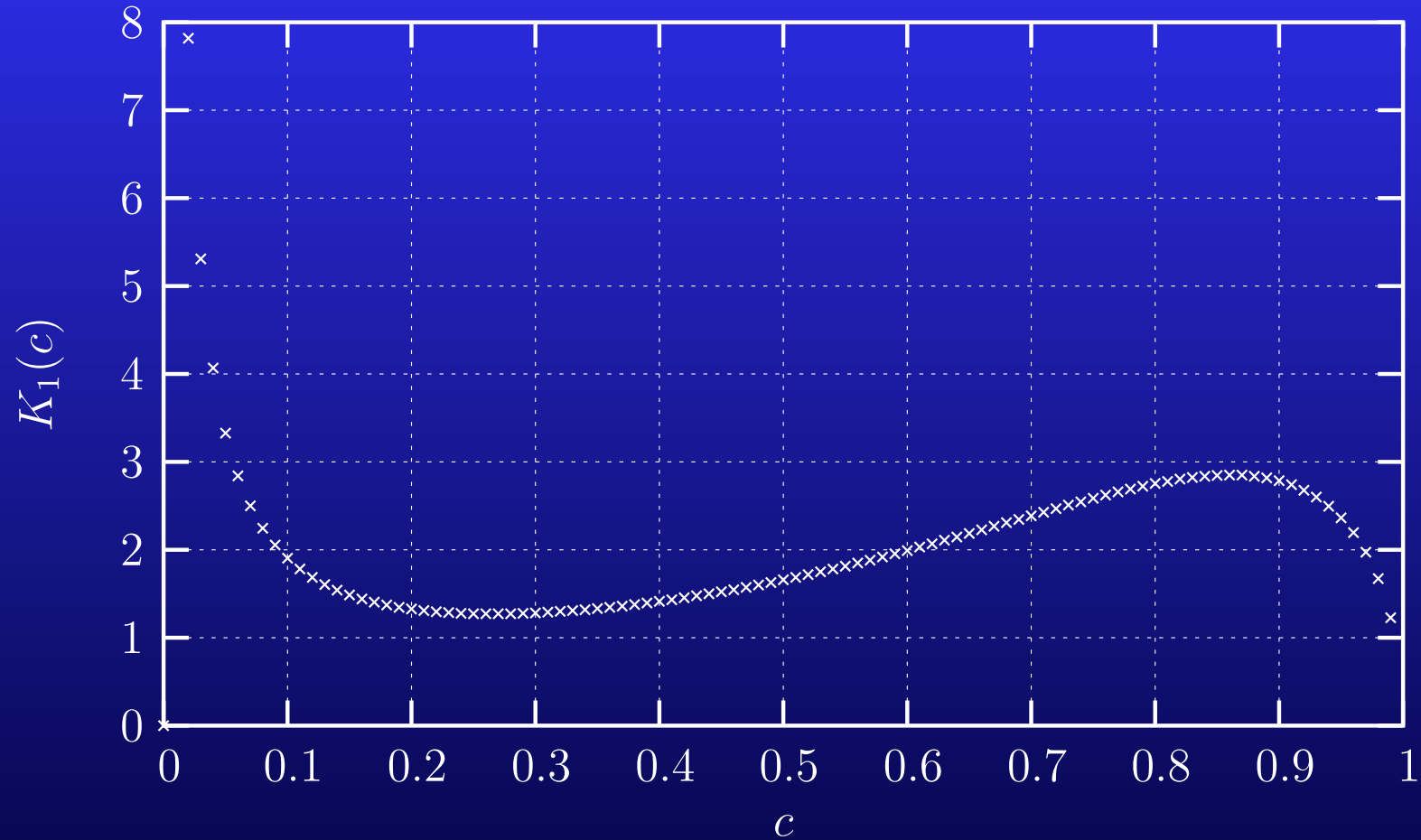
Speight's discretization





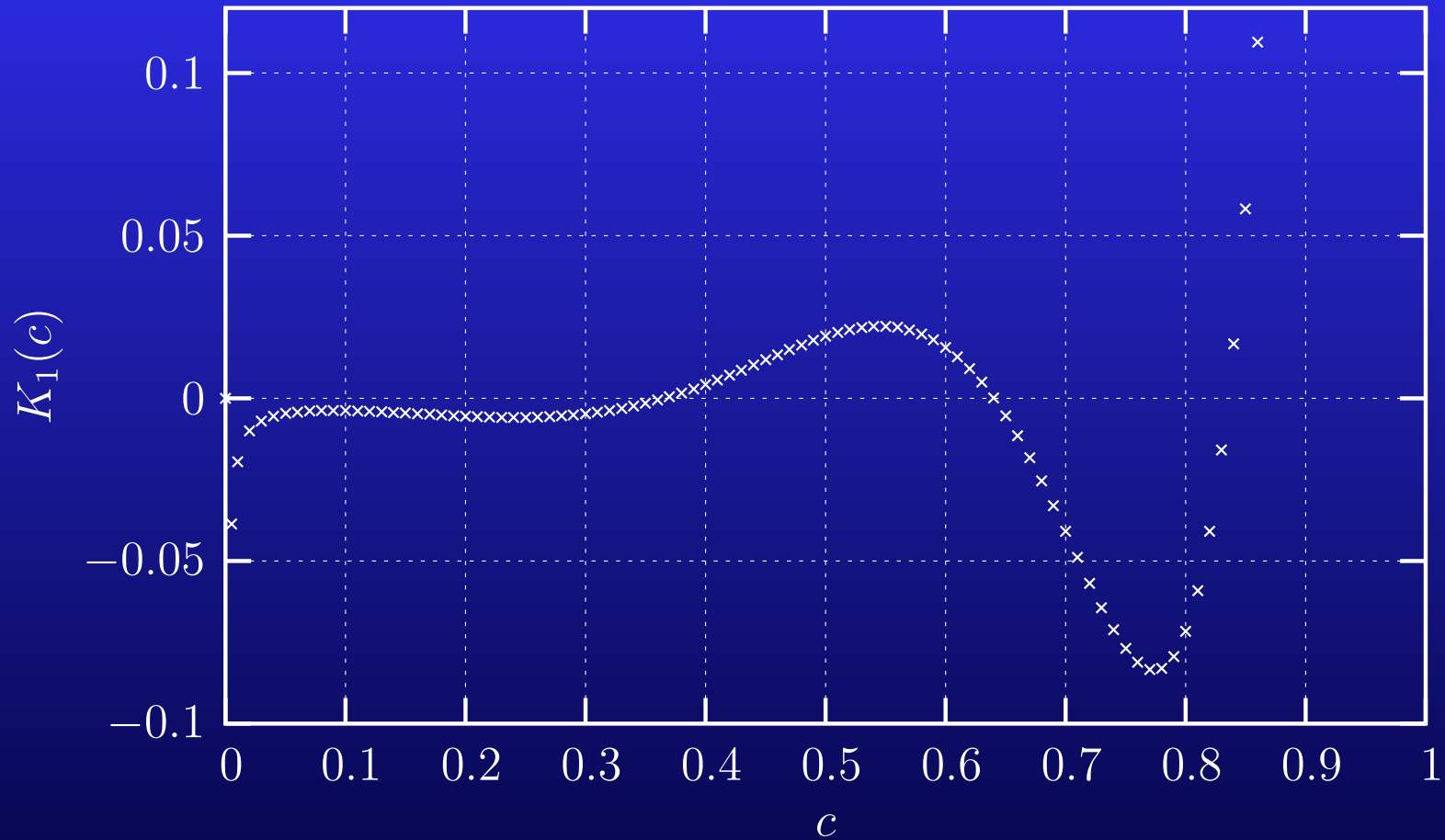
# Stokes Constant

Tovbis' discretization



# Stokes Constant

Kevrekidis' discretization

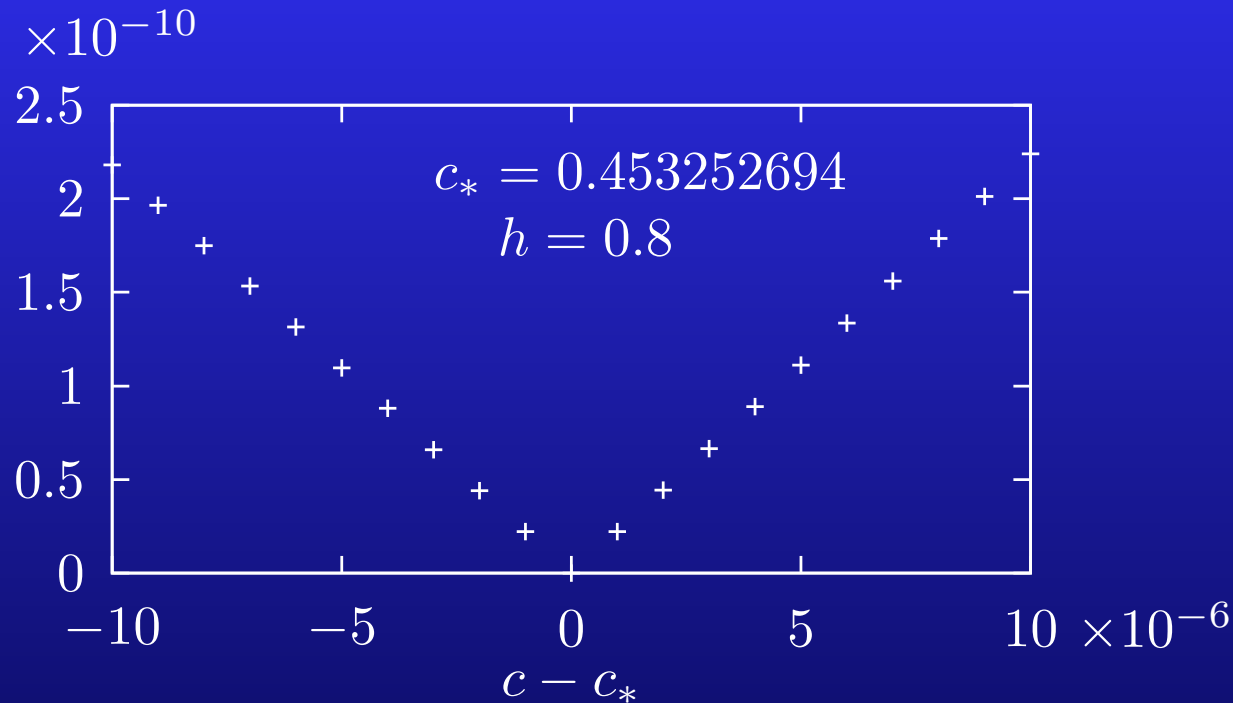


# Bifurcations of traveling kinks on $(c, h)$

- Zeros  $c = c_*$  of  $K_1(c)$  lie in the region  $c > 0.22$ , where only one pair  $p = \pm ik_1$  is purely imaginary.
- One can expect a bifurcation of the one-parameter curve on the  $(h, c)$  plane that passes through the point  $(0, c_*)$ .
- Numerical analysis of the bifurcation:
  - solve the differential advance-delay equation on  $z \in [-L, L]$  where  $L = 100$
  - subject to the anti-periodic boundary conditions  $\phi(L) = -\phi(-L)$
  - by using the iterative Newton's method with the continuum kink as starting guess,
  - the eight-order finite-difference approximation to the second derivative with the step size  $\Delta z = h/10$ .

# Numerical analysis of the bifurcation

## Speight's model



The average of  $[\phi(z) - \phi_{\text{ave}}]^2$  is computed over the the interval  $z \in [L - 20, L]$  for fixed values of parameter  $c$

# Bifurcation at $h = 0, c = 1$

Differential advance-delay equation (inner form):

$$c^2 \phi'' = \phi(\zeta+1) - 2\phi(\zeta) + \phi(\zeta-1) + h^2 \phi - h^2 Q(\phi(\zeta-1), \phi(\zeta), \phi(\zeta+1)),$$

where  $\zeta = z/h$ .

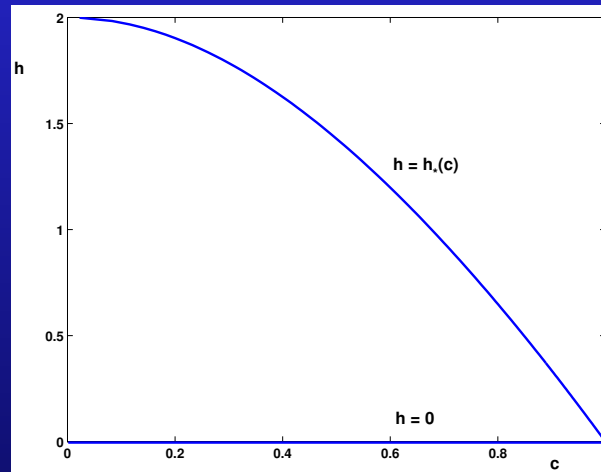
Near the point

$$h = 0, c = 1:$$

$$c^2 = 1 + \epsilon\gamma$$

$$h^2 = \epsilon^2\tau$$

$$\zeta_1 = \sqrt{\epsilon}\zeta$$



Truncated normal form for bifurcation:

$$\frac{1}{12} \phi^{(iv)}(\zeta_1) - \gamma \phi''(\zeta_1) + \tau \phi(\zeta_1)(1 - \phi^2(\zeta_1)) = 0.$$

# Numerical analysis of heteroclinic orbits

Truncated normal form:

$$\phi^{(\text{iv})} + \sigma\phi'' + \phi - \phi^3 = 0,$$

where  $\sigma = -\sqrt{12}\gamma/\sqrt{\tau}$  and  $\phi = \phi(t)$ .

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Linearization at  $\phi = \pm 1$  gives pairs of eigenvalues  $(\lambda_0, -\lambda_0)$  and  $(i\omega_0, -i\omega_0)$  with the one-dimensional unstable manifold:

$$\lim_{t \rightarrow -\infty} \phi_u(t) = -1, \quad \lim_{t \rightarrow -\infty} (\phi_u(t) + 1) e^{-\lambda_0 t} = C_0$$

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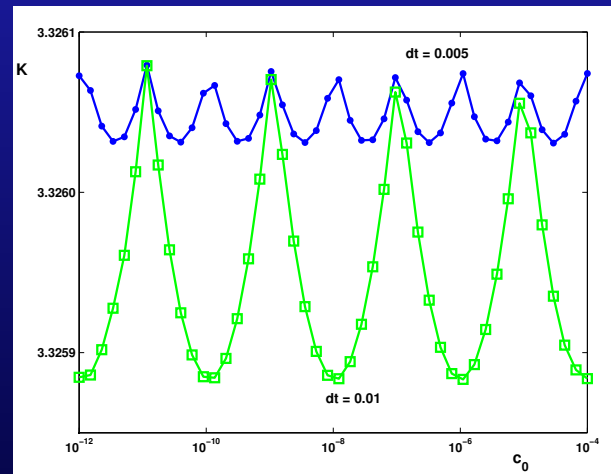
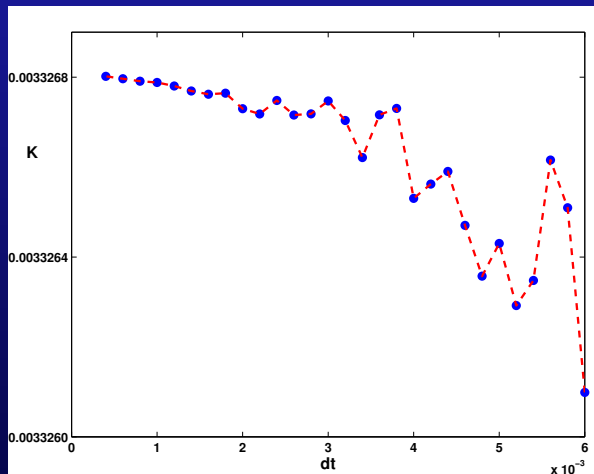
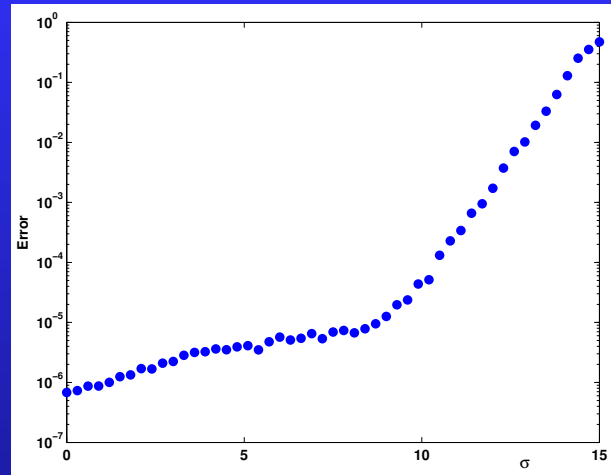
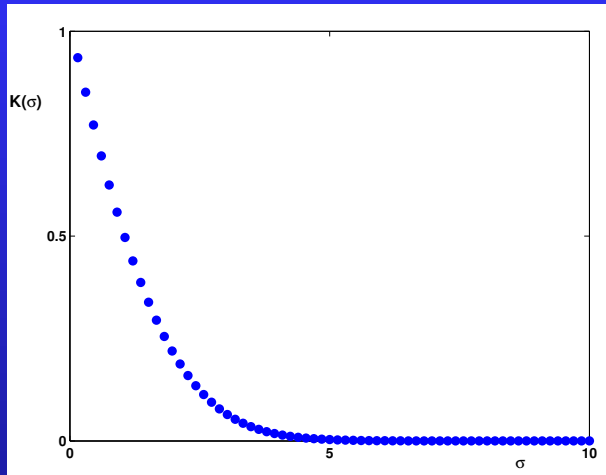
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The kink solution is odd in  $t \in \mathbb{R}$  (up to translational invariance). Iterating the initial-value problem along the unstable manifold from  $t = 0$  to  $t = t_0$  where  $\phi(t_0) = 0$ , one can compute the split function  $K(\sigma) = \phi''(t_0)$ , which may depend on numerical factors  $C_0$  and  $\Delta t$ .



# Numerical results on $K(\sigma)$



No bifurcations of kinks occur from the point  $h = 0, c = 1$ .

# Conclusions

- Existence of continuous stationary kinks at  $c = 0$  is not sufficient for existence of traveling kinks at  $c \neq 0$
- Bifurcations of traveling kinks may occur at isolated velocities with  $0 < c < 1$  (e.g. in the numerical analysis of the Speight's and Kevrekidis' exceptional nonlinearities)
- No bifurcations of traveling kinks occur from the point  $c = 1$
- It is problematic to consider asymptotic expansions in powers of  $c^2$  and the bifurcation of traveling kinks from the point  $c = 0$