Bifurcations of asymmetric vortices in symmetric harmonic potentials

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References:
C. Gallo & D.P., Asymptotic Analysis **73**, 53-96 (2011)
D.P., P. Kevrekidis, Nonlinearity **24**, 1271–1289 (2011)
D.P., P. Kevrekidis, preprint (2012)

Gross-Pitaevskii equation

Density waves in cigar–shaped Bose–Einstein condensates with repulsive inter-atomic interactions and a harmonic potential are modeled by the Gross-Pitaevskii equation

$$iv_\tau = -\frac{1}{2}\nabla_\xi^2 v + \frac{1}{2}|\xi|^2 v + |v|^2 v - \mu v,$$

where μ is the chemical potential, $\xi \in \mathbb{R}^d$, and ∇_ξ^2 is the Laplacian in ξ.

Using the scaling transformation,

$$v(\xi,t) = \mu^{1/2} u(x,t), \quad \xi = (2\mu)^{1/2} x, \quad \tau = 2t,$$

the Gross–Pitaevskii equation is transformed to the semi-classical form

$$i\,\varepsilon\, u_t + \varepsilon^2\, \nabla_x^2 u + (1 - |x|^2 - |u|^2)u = 0,$$

where $\varepsilon = (2\mu)^{-1}$ is a small parameter.

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Ground state

Limit $\mu \to \infty$ or $\varepsilon \to 0$ is referred to as the semi-classical or Thomas–Fermi limit. Physically, it is the limit of large density of the atomic cloud.

The ground state η_ε is the real positive solution of the stationary equation,

$$\varepsilon^2 \nabla_x^2 \eta_\varepsilon + (1 - |x|^2 - \eta_\varepsilon^2)\eta_\varepsilon = 0, \quad x \in \mathbb{R}^2.$$

Theorem (Ignat & Milot, 2006)

For any $\varepsilon \in \left(0, \frac{1}{2}\right)$, there exists a global minimizer of the Gross–Pitaevskii energy

$$E(u) = \int_{\mathbb{R}^2} \left(\varepsilon^2 |\nabla_x u|^2 + (|x|^2 - 1)|u|^2 + \frac{1}{2}|u|^4 \right) dx$$

in the energy space

$$X = \left\{ u \in H^1(\mathbb{R}^2) : \ |x|u \in L^2(\mathbb{R}^2) \right\}.$$

Ground state in the asymptotic theory

For small $\varepsilon > 0$, the ground state $\eta_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ decays to zero as $|x| \to \infty$ faster than any exponential function

$$0 < \eta_\varepsilon(x) \leq C\,\varepsilon^{1/3} \exp\left(\frac{1-|x|^2}{4\,\varepsilon^{2/3}}\right), \quad \text{for all } |x| \geq 1.$$

The Thomas–Fermi approximation is

$$\eta_0(x) := \lim_{\varepsilon \to 0} \eta_\varepsilon(x) = \begin{cases} (1-|x|^2)^{1/2}, & \text{for } |x| < 1, \\ 0, & \text{for } |x| > 1, \end{cases}$$

Theorem (Gallo & P., 2011)

For sufficiently small $\varepsilon > 0$, there is $C > 0$ such that

$$\|\eta_\varepsilon - \eta_0\|_{L^\infty} \leq C\,\varepsilon^{1/3}, \quad \|\nabla_x \eta_\varepsilon\|_{L^\infty} \leq C\,\varepsilon^{-1/3}.$$

Vortices

The vortex u_ε is a complex-valued solution of the stationary equation,

$$\varepsilon^2 \nabla_x^2 u_\varepsilon + (1 - |x|^2 - |u_\varepsilon|^2) u_\varepsilon = 0, \quad x \in \mathbb{R}^2.$$

The product representation

$$u(x,t) = \eta_\varepsilon(|x|) v(x,t)$$

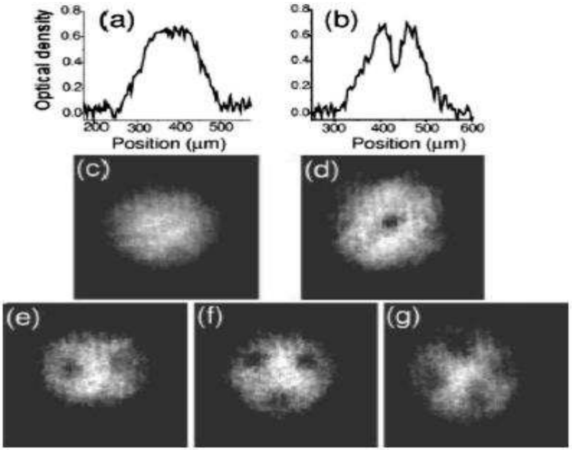
brings the Gross–Pitaevskii equation to the equivalent form

$$i \varepsilon \eta_\varepsilon^2 v_t + \varepsilon^2 \nabla_x \left(\eta_\varepsilon^2 \nabla_x v \right) + \eta_\varepsilon^4 (1 - |v|^2) v = 0,$$

where $\lim_{|x| \to \infty} |v(x)| = 1$.

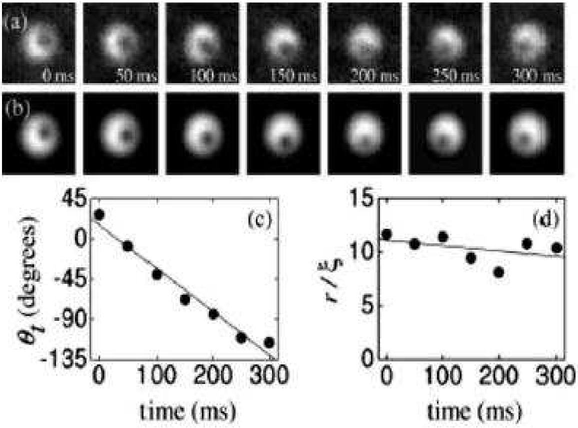
Symmetric vortex of charge $m \in \mathbb{N}$ corresponds to the choice $v = \psi(r/\varepsilon) e^{im\theta}$, where (r,θ) are polar coordinates on $\mathbb{R}^2$ and $\psi(r/\varepsilon) \to 1$ as $r \to \infty$.

Experimental studies of vortices



Absorption images of a BEC stirred with a laser beam.
From Madison et al., 2000.

Experimental studies of vortex precession



Vortex precession in a trapped two-component BEC.
From Anderson et al., 2000.

Theoretical studies of vortices

Earlier results in physics literature:

- Castin & Dum (1999) and Aftalion & Du (2001) - rotating vortices can become local and later global minimizers of energy for larger frequencies
- Fetter & Svidzinksy (2001) - vortex configurations can be understood through effective energy
- Möttönen *et al.* (2005) - computations of the interaction energy for two and four vortices; prediction of stationary dipoles and quadrupoles
- Li *et al.* (2008) - dynamics of a vortex–antivortex pair on a phase plane
- Middelkamp *et al.* (2010) - numerical computations of eigenvalues for single vortices, dipoles and quadrupoles by using relaxation methods
- Pelinovsky & Kevrekidis (2011) - variational approximations of eigenvalues for single vortices, dipoles and quadrupoles by the Rayleight–Ritz method
- Kollar & Pego (2012) - numerical computations of eigenvalues for charge-one and charge-two vortices by using Evans functions

Main results I

Theorem (P. & Kevrekidis, 2012)

For every $\epsilon \in \left(0, \frac{1}{4}\right)$, there exists a unique classical solution $u = \psi_\epsilon(r)e^{i\theta}$ for the symmetric vortex of charge one.

For small $|\epsilon - \frac{1}{4}|$, the symmetric vortex is spectrally stable in the sense that all eigenvalues of the spectral stability problem are purely imaginary and semi-simple, except for the double zero eigenvalue.

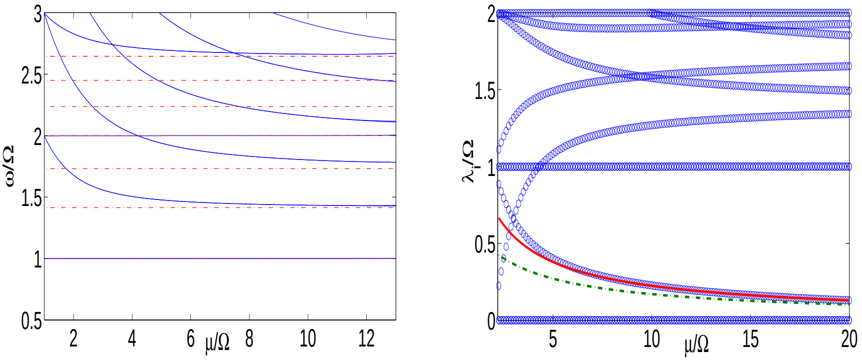
The symmetric vortex is a saddle point of the Gross–Pitaevskii energy

$$E(u) = \int_{\mathbb{R}^2} \left(\varepsilon^2 |\nabla u|^2 + (|x|^2 - 1)|u|^2 + \frac{1}{2}|u|^4 \right) dx$$

but it is a global minimizer of the reduced energy ($u = \psi(r)e^{i\theta}$)

$$E_1(\psi) = \int_0^\infty \left[\epsilon^2 \left(\frac{d\psi}{dr} \right)^2 + \frac{\epsilon^2 \psi^2}{r^2} + (r^2 - 1)\psi^2 + \frac{1}{2}\psi^4 \right] rdr.$$

Spectral stability of symmetric charge-one vortices



Stable (purely imaginary) eigenvalues of the spectral stability problem for the ground state η_ε (left) and the symmetric vortex $\psi(r)e^{i\theta}$ (right).
From Pelinovsky & Kevrekidis (2011).

Main results II

Theorem (P. & Kevrekidis, 2012)

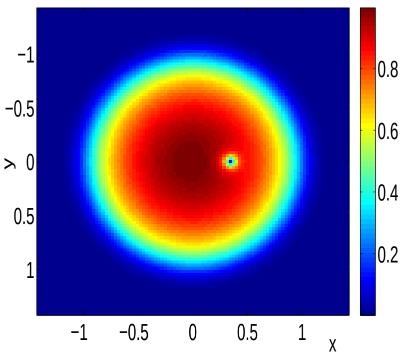
For every $\epsilon \in \left(0, \frac{1}{2}\right)$ with small $|\epsilon - \frac{1}{2}|$, there is a rotational frequency $\omega_0 \in (0, 2)$ such that for every $\omega > \omega_0$ with small $|\omega - \omega_0|$, besides the symmetric vortex $u = \psi(r)e^{i\theta}$, there exists an asymmetric vortex solution $u = u_\alpha(x, y)$ of the Gross–Pitaevskii equation with a rotational term.

The center of $|u_\alpha|$ is placed on the circle of radius $|a|$ centered at the origin $(0, 0) \in \mathbb{R}^2$ at an arbitrary angle α. There is $C > 0$ such that

$$|a| \leq C\sqrt{\epsilon(\omega - \omega_0)}.$$

For $\omega > \omega_0$, the symmetric vortex is a local minimizer of the energy $E(u)$, whereas the asymmetric vortex is a saddle point of the energy $E(u)$.

Steady precession of asymmetric charge-one vortices



Spatial contour plots of the amplitude (left) and phase (right) of a rotating charge-one vortex. From Pelinovsky & Kevrekidis (2012).

Main results III

Theorem (P. & Kevrekidis, 2012)

For every $\epsilon \in \left(0, \frac{1}{2}\right)$ and $\omega > \omega_0$ with small $|\epsilon - \frac{1}{2}|$ and $|\omega - \omega_0|$, the symmetric vortex of charge one is orbitally stable in the following sense: for any $\sigma > 0$ there is a $\delta > 0$, such that if $\|u(0) - \psi(r)e^{i\theta}\|_X \leq \delta$, then

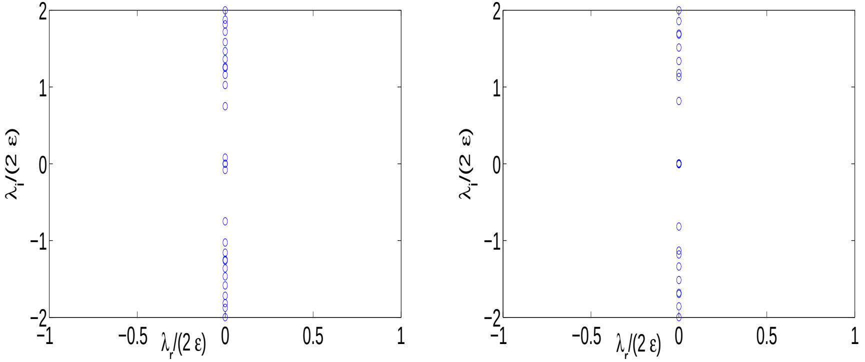
$$\inf_{\beta \in \mathbb{R}} \|u(t) - e^{i\beta}\psi(r)e^{i\theta}\|_X \leq \sigma, \quad t \in \mathbb{R}_+,$$

At the same time, the asymmetric vortex is also orbitally stable in the following sense: for any $\sigma > 0$ there is a $\delta > 0$, such that if $\|u(0) - u_0\|_X \leq \delta$, then

$$\inf_{(\beta,\alpha) \in \mathbb{R}^2} \|u(t) - e^{i\beta}u_\alpha\|_X \leq \sigma, \quad t \in \mathbb{R}_+.$$

Here $X = \left\{u \in H^1(\mathbb{R}^2) : \ |x|u \in L^2(\mathbb{R}^2)\right\}$ is the energy space of the Gross–Pitaevskii equation.

Spectral stability of rotating charge-one vortices



Left: eigenvalues of the spectral stability problem for the symmetric vortex $\psi(r)e^{i\theta}$. Right: eigenvalues of the spectral stability problem for the asymmetric vortex u_α. From Pelinovsky & Kevrekidis (2012).

Variational construction of vortices

The equivalent Gross–Pitaevskii equation

$$i\,\varepsilon\,\eta_\varepsilon^2 v_t + \varepsilon^2\,\nabla_x\left(\eta_\varepsilon^2 \nabla_x v\right) + \eta_\varepsilon^4 (1 - |v|^2) v = 0,$$

is the Euler–Lagrange equation for the Lagrangian $L(v) = K(v) + E(v)$ with the kinetic energy

$$K(v) = \frac{i}{2}\,\varepsilon \int_{\mathbb{R}^2} \eta_\varepsilon^2 (v \bar{v}_t - \bar{v} v_t) dx$$

and the potential energy

$$E(v) = \varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2 |\nabla_x v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \eta_\varepsilon^4 (1 - |v|^2)^2 dx.$$

Substituting a vortex ansatz

$$v(x - x_0, y - y_0) = \Psi(R) e^{i\theta}, \quad R = \frac{r}{\varepsilon}$$

and computing Euler–Lagrange equations for parameters (x_0, y_0) yield the system of differential equations.

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Approximate energy for the symmetric vortex

Proposition (P. & Kevrekidis, 2011)

For small $\varepsilon > 0$ and small $(x_0, y_0) \in \mathbb{R}^2$, the kinetic and potential energies of a single vortex is approximated by

$$K(v) = \pi\,\varepsilon(x_0\dot{y}_0 - y_0\dot{x}_0)\left(1 + \mathcal{O}(\varepsilon) + \mathcal{O}(x_0^2 + y_0^2)\right)$$

and

$$E(v) - E(\psi e^{i\theta}) = -\pi\,\varepsilon\,\omega_a(x_0^2 + y_0^2)\left(1 + \mathcal{O}(\varepsilon^{1/3}) + \mathcal{O}(x_0^2 + y_0^2)\right),$$

where ω_a is given by

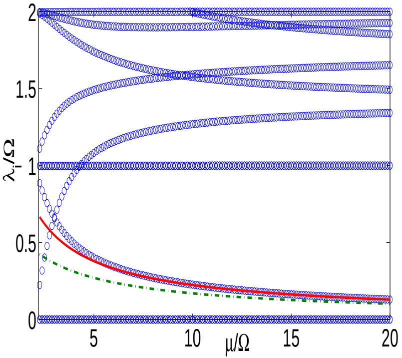
$$\omega_a = \varepsilon\left[2\log(1/\varepsilon) + 1 + 2\int_0^\infty \left[\left(\frac{d\Psi}{dR}\right)^2 + \frac{1}{R^2}\left(\Psi^2 - \frac{R^2}{1+R^2}\right)\right] R dR\right].$$

Eigenfrequencies of the charge-one vortex

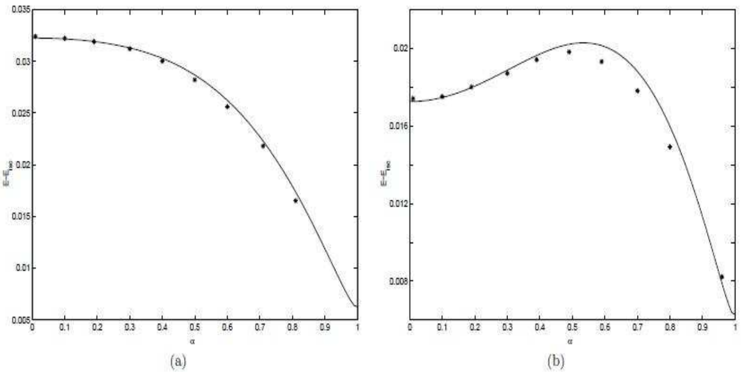
Euler–Lagrange equations for the leading part of $L(v) = K(v) + E(v)$ give

$$-\dot{x}_0 = \omega_a y_0, \quad \dot{y}_0 = \omega_a x_0,$$

where $\omega_a = 2\,\varepsilon\,|\log(\varepsilon)| + \mathcal{O}(\varepsilon)$.



Approximate energy in the case of rotation



Self-energy of a vortex of charge one as a function of the distance of the core from the trap center, for two values of the rotational frequency ω.
From Castin & Dum (1999)

Other variational and rigorous results

Recall the variational approximation of the eigenvalue

$$\omega_a = 2\,\varepsilon\,|\log(\varepsilon)| + \mathcal{O}(\varepsilon),$$

which corresponds to a periodic precession of the symmetric vortex of charge one around the origin $(0,0) \in \mathbb{R}^2$ with a small displacement from the origin.

- Castin & Dum (1999): the symmetric vortex becomes a local minimizer for $\omega > \omega_a$ and a global minimizer of energy for $\omega > 2\omega_a$
- Aftalion & Du (2001), Ignat & Millot (2006) : a vortex of charge one is a global minimizer of energy for sufficiently large $\omega > 2\omega_a$ and its core is located close to the trap center.

Our task: to prove existence of the local bifurcation at $\omega = \omega_a$ and to distinguish between symmetric and asymmetric vortices of charge one.

Steadily rotating vortices

In the rotating coordinate frame,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad \omega \in \mathbb{R},$$

the Gross–Pitaevskii equation takes the form,

$$i\,\varepsilon\, u_t + \varepsilon^2 (u_{\xi\xi} + u_{\eta\eta}) + (1 - \xi^2 - \eta^2 - |u|^2) u - i\,\varepsilon\,\omega (\xi u_\eta - \eta u_\xi) = 0.$$

The symmetric vortex of charge one is given by

$$u(\xi,\eta) = \sqrt{1+\varepsilon\,\omega}\,\psi_\nu(r) e^{i\theta}, \quad \sqrt{\xi^2+\eta^2} = \sqrt{1+\varepsilon\,\omega}\, r,$$

where $\psi_\nu(r) > 0$ satisfies

$$\nu^2 \left(\frac{d^2\psi_\nu}{dr^2} + \frac{1}{r}\frac{d\psi_\nu}{dr} - \frac{\psi_\nu}{r^2} \right) + (1 - r^2 - \psi_\nu^2)\psi_\nu = 0, \quad \nu = \frac{\varepsilon}{1+\varepsilon\,\omega}.$$

Existence of symmetric vortex

Schrödinger operator for the quantum harmonic oscillator

$$H(\nu) := -\nu^2(\partial_x^2+\partial_y^2)+x^2+y^2-1, \quad \mathrm{D}(H(\nu)) := \left\{u \in H^2(\mathbb{R}^2) : \quad |x|^2 u \in L^2(\mathbb{R}^2)\right\}.$$

The spectrum of $H(\nu)$ in $L^2(\mathbb{R}^2)$ is purely discrete:

$$\sigma(H(\nu)) = \left\{\lambda_{n,m}(\nu) = -1 + 2\nu(n+m+1), \quad (n,m) \in \mathbb{N}_0^2\right\},$$

- $\nu = \frac{1}{2}$ - bifurcation of a ground state $\eta_\nu(r)$ $(n = m = 0)$ existing in $(0, \frac{1}{2})$
- $\nu = \frac{1}{4}$ - bifurcation of a charge-one vortex $\psi_\epsilon(r)e^{i\theta}$ $(n + m = 1)$ in $(0, \frac{1}{4})$

Lemma

Let $\mu := \frac{1}{16} - \nu^2$ *and* $\psi_0(r) = re^{-2r^2}$. *Then,*

$$\sup_{r\in\mathbb{R}_+} \left|\psi_\nu(r) - (128\mu)^{1/2}\psi_0(r)\right| \le C\mu^{3/2}.$$

Energy of the symmetric vortex

Substituting

$$u(x,y) = \psi_\nu(r)e^{i\theta} + U(x,y)$$

to the energy functional $E(u)$, we obtain

$$E(u) - E(\psi_\nu e^{i\theta}) = \langle \mathbf{U}, \mathcal{H}(\nu)\mathbf{U}\rangle_{L^2} + \mathcal{O}(\|\mathbf{U}\|^3_{H^1}), \quad (1)$$

where $\mathbf{U} = [U, \bar{U}]^T$. Using the decomposition in normal modes,

$$U(x,y) = \sum_{m\in\mathbb{Z}} V_m(r)e^{im\theta}, \quad \bar{U}(x,y) = \sum_{m\in\mathbb{Z}} W_m(r)e^{im\theta},$$

we obtain an uncoupled eigenvalue problem for components (V_m, W_{m-2}):

$$H_m(\nu)\left[\begin{array}{c} V_m \\ W_{m-2}\end{array}\right] = \nu\lambda \left[\begin{array}{c} V_m \\ W_{m-2}\end{array}\right], \quad m \in \mathbb{Z},$$

where

$$H_m(\nu) = \left[\begin{array}{cc} -\nu^2\Delta_m + r^2 - 1 + 2\psi_\nu^2 & \psi_\nu^2 \\ \psi_\nu^2 & -\nu^2\Delta_{m-2} + r^2 - 1 + 2\psi_\nu^2 \end{array}\right].$$

Symmetric vortex as a saddle point of energy

Recall

$$H_m(\nu) = \begin{bmatrix} -\nu^2\Delta_m + r^2 - 1 + 2\psi_\nu^2 & \psi_\nu^2 \\ \psi_\nu^2 & -\nu^2\Delta_{m-2} + r^2 - 1 + 2\psi_\nu^2 \end{bmatrix}.$$

and

$$\sigma(-\nu^2\Delta_m + r^2 - 1) = \{\lambda_{n,m}(\nu) = -1 + 2\nu(2n + m + 1), \quad n \in \mathbb{N}_0\}.$$

Lemma

For $\nu < \frac{1}{4}$ with small $|\nu - \frac{1}{4}|$, there exists exactly one negative eigenvalue $\lambda_0(\nu)$, which has algebraic multiplicity two and is associated to the eigenvectors of $H_2(\nu)$ and $H_0(\nu)$. Moreover, λ_0 is a C^1 function of ν satisfying

$$\lim_{\nu \uparrow \frac{1}{4}} \lambda_0(\nu) = -2.$$

The zero eigenvalue of $H_1(\nu)$ is simple and is associated with the gauge symmetry of the Gross-Pitaevskii equation.
All other eigenvalues of $H_m(\nu)$ are strictly positive.

Spectral stability of symmetric vortex

Non-self-adjoint spectral problem:

$$H_m(\nu) \begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix} = \nu\lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix}, \quad m \in \mathbb{Z}.$$

Lemma

For $\nu < \frac{1}{4}$ with small $|\nu - \frac{1}{4}|$, the spectral problem admits only real eigenvalues λ of equal algebraic and geometric multiplicities, in addition to the double zero eigenvalue for $m = 1$.
The smallest nonzero eigenvalue for $m = 2$ is $\lambda = +\omega_0(\nu)$ and for $m = 0$ is $\lambda = -\omega_0(\nu)$, where $\omega_0(\nu) > 0$ and $\lim_{\nu \uparrow \frac{1}{4}} \omega_0(\nu) = 2$. These eigenvalues are simple and correspond to the eigenvectors $\mathbf{V}_\pm(\nu)$ such that

$$\langle \mathbf{V}_+(\nu), H_2(\nu)\mathbf{V}_+(\nu)\rangle_{L^2_r} = \langle \mathbf{V}_-(\nu), H_0(\nu)\mathbf{V}_-(\nu)\rangle_{L^2_r} < 0.$$

The quadratic form associated with operators $H_m(\nu)$ is strictly positive for the eigenvectors corresponding to any other eigenvalue of the spectral problems.

Two linearizations in the rotational case

If we substitute $u(\xi,\eta,t) = \psi_\nu(r)e^{i\theta} + U(\xi,\eta,t)$ to the Gross–Pitaevskii equation with the rotation and adopt the decomposition

$$U(\xi,\eta,t) = \sum_{m\in\mathbb{Z}} V^{(m)}(\rho)e^{im\theta}e^{-i\sigma t}, \quad \bar{U}(\xi,\eta,t) = \sum_{m\in\mathbb{Z}} W^{(m)}(\rho)e^{im\theta}e^{-i\sigma t},$$

then we end up with the spectral stability problem

$$H_\omega^{(m)} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix} = \varepsilon\,\sigma \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix},$$

where

$$H_\omega^{(m)} = \begin{bmatrix} 1-\rho^2+\varepsilon^2\Delta_m+\varepsilon\,\omega m-2\psi^2 & -\psi^2 \\ -\psi^2 & 1-\rho^2+\varepsilon^2\Delta_{m-2}-\varepsilon\,\omega(m-2)-2\psi^2 \end{bmatrix}$$

On the other hand, linearization of the stationary problem is related to the spectrum of the self-adjoint eigenvalue problem

$$H_\omega^{(m)} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix} = \varepsilon\,\lambda \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix}.$$

Zero eigenvalue results in a bifurcation of stationary vortices.

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Zero eigenvalue results in a bifurcation of stationary vortices.

Transformation

Adopting new variables $\rho = \sqrt{1+\varepsilon\,\omega} r$ and $\nu = \varepsilon\,/(1+\varepsilon\,\omega)$, we transform the spectral stability problem to the form,

$$H_m(\nu)\left[\begin{array}{c} V_m \\ W_{m-2} \end{array}\right] = \nu(\sigma + \omega(m-1))\left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]\left[\begin{array}{c} V_m \\ W_{m-2} \end{array}\right], \quad m \in \mathbb{Z},$$

and the self-adjoint eigenvalue problem to the form,

$$H_m(\nu)\left[\begin{array}{c} V_m \\ W_{m-2} \end{array}\right] = \nu\lambda\left[\begin{array}{c} V_m \\ W_{m-2} \end{array}\right] + \nu\omega(m-1)\left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]\left[\begin{array}{c} V_m \\ W_{m-2} \end{array}\right],$$

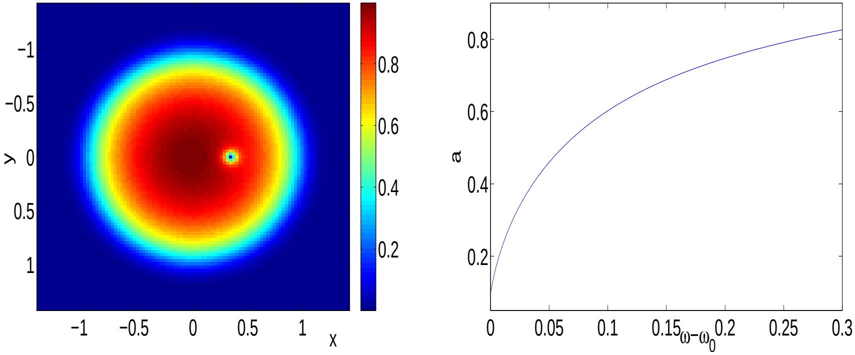
- The symmetric vortex is spectrally stable for

$$\nu \in (\nu_0, \frac{1}{4}) \quad \Leftrightarrow \quad \epsilon \in \left(\frac{\epsilon_0}{1-\epsilon_0\omega}, \frac{1}{4-\omega}\right).$$

- There exists a bifurcation of the symmetric vortex for $m = 2$ and $\omega = \omega_0(\nu) \in (0,2)$ corresponding to $\epsilon < \frac{1}{2}$. Moreover, if $\lambda(\omega)$ is the eigenvalue such that $\lambda(\omega_0) = 0$, then $\lambda'(\omega_0) > 0$.

Bifurcation of asymmetric vortices

Rotating vortex is born via the supercritical pitchfork bifurcation with radial symmetry for $\omega > \omega_0$. Its center is placed at a point on the circle of radius *a* on the (ξ, η)-plane, where $a \sim \sqrt{\varepsilon(\omega - \omega_0)}$.



Lyapunov–Schmidt reductions I

Root finding problem for $N(u;\omega) : H^2(\mathbb{R}^2) \times \mathbb{R} \to L^2(\mathbb{R}^2)$:

$$N(u;\omega) := -\epsilon^2(u_{\xi\xi} + u_{\eta\eta}) + (\xi^2 + \eta^2 - 1 + |u|^2)u + i\epsilon\omega(\xi u_\eta - \eta u_\xi).$$

The kernel of linearization at the bifurcation point:

$$\mathrm{Ker}(D_u N(\psi_0 e^{i\theta};\omega_0)) = \mathrm{span}\left\{ \left[\begin{array}{c} \psi_0(r)e^{i\theta} \\ -\psi_0(r)e^{-i\theta} \end{array} \right], \left[\begin{array}{c} V_2(r)e^{2i\theta} \\ W_0(r) \end{array} \right], \left[\begin{array}{c} W_0(r) \\ V_2(r)e^{-2i\theta} \end{array} \right] \right\}$$

Decomposition

$$u = \psi_0(r)e^{i\theta} + aV_2(r)e^{2i\theta} + \bar{a}W_0(r) + U, \quad \omega = \omega_0 + \Omega.$$

Lyapunov–Schmidt reductions II

After near-identity transformations to eliminate quadratic terms in a, we obtain the normal form equation

$$a\left(2\epsilon\Omega\sigma + \beta|a|^2 + \mathcal{O}(|a|^4)\right) = 0,$$

where

$$2\sigma = \lambda'(\omega_0)\left(\|V_2\|^2_{L^2_r} + \|W_0\|^2_{L^2_r}\right) > 0$$

and

$$\beta = -\frac{1}{512} + \mathcal{O}(\mu), \quad \mu := \frac{1}{16} - \nu^2 > 0.$$

For small μ, we have the supercritical pitchfork bifurcation with radial symmetry:

$$|a|^2 = 32\epsilon(\omega-\omega_0) + \mathcal{O}((\omega-\omega_0)^2, \mu),$$

and $\alpha = \arg(a)$ is an arbitrary angle in the (ξ, η)-plane for the vortex core on the circle of radius $|a|$.

Stability of vortices

Symmetric vortex of charge one $\psi(r)e^{i\theta}$ is a local minimizer of energy $E(u)$ in space X for $\omega > \omega_0$. Therefore, it is orbitally stable in the sense: for any $\sigma > 0$ there is a $\delta > 0$, such that if $\|u(0) - \psi(r)e^{i\theta}\|_X \leq \delta$, then

$$\inf_{\beta \in \mathbb{R}} \|u(t) - e^{i\beta}\psi(r)e^{i\theta}\|_X \leq \sigma, \quad t \in \mathbb{R}_+,$$

The new asymmetric vortex u_α is a saddle point of energy $E(u)$ in space X. The linearization operator $D_u N(u_\alpha; \omega)$ has exactly one negative eigenvalue and the two-dimensional kernel:

$$\mathrm{Ker}(D_u N(u_\alpha; \omega)) = \mathrm{span}\left\{ \begin{bmatrix} u_\alpha \\ -\bar{u}_\alpha \end{bmatrix}, \begin{bmatrix} \partial_\alpha u_\alpha \\ \partial_\alpha \bar{u}_\alpha \end{bmatrix} \right\}.$$

We show that this vortex is also orbitally stable in the sense: for any $\sigma > 0$ there is a $\delta > 0$, such that if $\|u(0) - u_0\|_X \leq \delta$, then

$$\inf_{(\theta,\alpha) \in \mathbb{R}^2} \|u(t) - e^{i\theta}u_\alpha\|_X \leq \sigma, \quad t \in \mathbb{R}_+.$$

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Stability of the asymmetric vortex

We need to prove that the linearization operator $D_u N(u_\alpha;\omega)$ is non-negative in the constrained space

$$L^2_c(\mathbb{R}^2) = \left\{ U \in L^2(\mathbb{R}^2) : \quad \langle \mathbf{V}, \sigma_3 \mathbf{U} \rangle := \int_{\mathbb{R}^2} \left(\bar{V} U - \bar{W} \bar{U} \right) dx = 0, \right.$$
$$\left. \text{for every } \mathbf{V} = \begin{bmatrix} V \\ W \end{bmatrix} \in \mathrm{Ker}(D_u N(u_\alpha;\omega)) \right\},$$

where $\sigma_3 = \mathrm{diag}(1,-1)$ is due to the symplectic structure of the Gross-Pitaevskii equation.

This result is equivalent to the fact that the matrix of symplectic projections

$$\begin{bmatrix} \langle \mathbf{V}_g, \sigma_3 \tilde{\mathbf{V}}_g \rangle & \langle \mathbf{V}_r, \sigma_3 \tilde{\mathbf{V}}_g \rangle \\ \langle \mathbf{V}_g, \sigma_3 \tilde{\mathbf{V}}_r \rangle & \langle \mathbf{V}_r, \sigma_3 \tilde{\mathbf{V}}_r \rangle \end{bmatrix}$$

has exactly one negative eigenvalue, where

$$\tilde{\mathbf{V}}_g = \mathcal{H}^{-1} \begin{bmatrix} u_\alpha \\ \bar{u}_\alpha \end{bmatrix}, \quad \tilde{\mathbf{V}}_r = \mathcal{H}^{-1} \begin{bmatrix} \partial_\alpha u_\alpha \\ -\partial_\alpha \bar{u}_\alpha \end{bmatrix}.$$

are generalized eigenvectors of the generalized kernel of $D_u N(u_\alpha;\omega)$.

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Conclusion and open questions

We have described the local bifurcation results for the birth of stable rotating asymmetric vortices of charge one in the Gross-Pitaevskii equation with a symmetric harmonic potential.

For supercritical rotational frequency, symmetric vortices of charge one are local minimizers of energy and asymmetric vortices of charge one are saddle points of the energy. Nevertheless, both vortices are orbitally stable in the time-dependent perturbations.

Open question: Can these results be extended in the entire existence interval $\left(0, \frac{1}{4}\right)$ (in terms of parameter ν)? In particular, can these results be proven in the Thomas–Fermi limit $\nu \to 0$ ($\epsilon \to 0$)?