Bifurcations of asymmetric vortices in symmetric harmonic potentials

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References:

D.P., P. Kevrekidis, Nonlinearity **24**, 1271–1289 (2011)
D.P., P. Kevrekidis, AMRE **2013**, 127–164 (2013)

Gross-Pitaevskii equation

Density waves in cigar–shaped Bose–Einstein condensates with repulsive inter-atomic interactions placed in a magnetic trap are modeled by the Gross-Pitaevskii equation with the harmonic potential

$$i v_\tau = -\frac{1}{2}\nabla_\xi^2 v + \frac{1}{2}|\xi|^2 v + |v|^2 v - \mu v,$$

where μ is the chemical potential, $\xi \in \mathbb{R}^d$, and ∇_ξ^2 is the Laplacian in ξ.

Using the scaling transformation,

$$v(\xi, t) = \mu^{1/2} u(x, t), \quad \xi = (2\mu)^{1/2} x, \quad \tau = 2t,$$

the Gross–Pitaevskii equation is transformed to the semi-classical form

$$i\,\varepsilon\, u_t + \varepsilon^2\, \nabla_x^2 u + (1 - |x|^2 - |u|^2) u = 0,$$

where $\varepsilon = (2\mu)^{-1}$ is a small parameter.

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Ground (vortex-free) state

Limit $\mu \to \infty$ or $\varepsilon \to 0$ is referred to as the semi-classical or Thomas–Fermi limit. Physically, it is the limit of large density of the atomic cloud.

The ground state η_ε is the real positive solution of the stationary equation,

$$\varepsilon^2 \nabla_x^2 \eta_\varepsilon + (1 - |x|^2 - \eta_\varepsilon^2)\eta_\varepsilon = 0, \quad x \in \mathbb{R}^2.$$

For small $\varepsilon > 0$, the ground state $\eta_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^2)$ decays to zero as $|x| \to \infty$ faster than any exponential function

$$0 < \eta_\varepsilon(x) \leq C\,\varepsilon^{1/3} \exp\left(\frac{1 - |x|^2}{4\,\varepsilon^{2/3}}\right), \quad \text{for all } \; |x| \geq 1.$$

The Thomas–Fermi approximation

$$\eta_0(x) := \lim_{\varepsilon \to 0} \eta_\varepsilon(x) = \begin{cases} (1 - |x|^2)^{1/2}, & \text{for } \; |x| < 1, \\ 0, & \text{for } \; |x| > 1, \end{cases}$$

was justified by Ignat–Milot (2006); Gallo–Pelinovsky (2011).

Vortex states

The static vortex u_ε is a complex-valued solution of the stationary equation,

$$\varepsilon^2 \nabla_x^2 u_\varepsilon + (1 - |x|^2 - |u_\varepsilon|^2) u_\varepsilon = 0, \quad x \in \mathbb{R}^2.$$

The product representation

$$u(x,t) = \eta_\varepsilon(x) v(x,t)$$

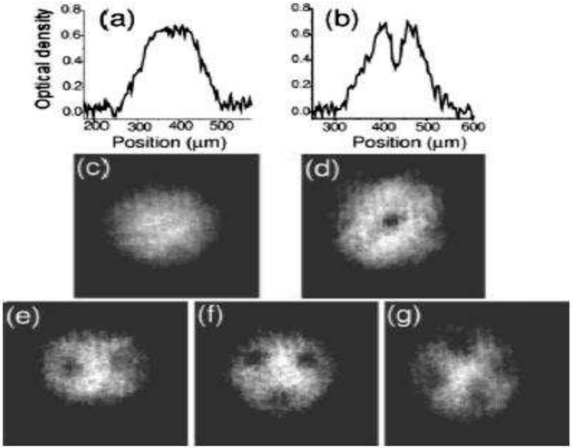
brings the Gross–Pitaevskii equation to the equivalent form

$$i \varepsilon \eta_\varepsilon^2 v_t + \varepsilon^2 \nabla_x \left(\eta_\varepsilon^2 \nabla_x v \right) + \eta_\varepsilon^4 (1 - |v|^2) v = 0,$$

where $\lim_{|x| \to \infty} |v(x)| = 1$.

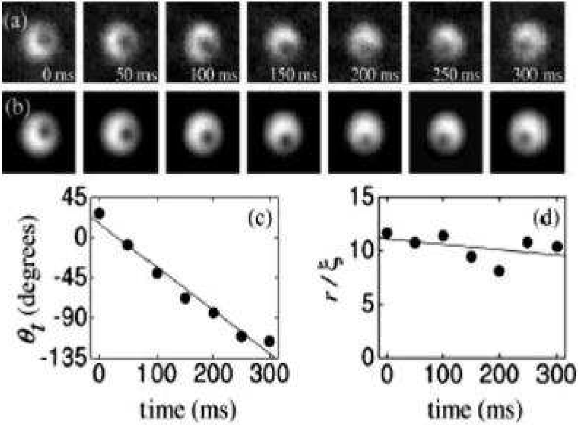
Symmetric vortex of charge $m \in \mathbb{N}$ corresponds to the choice $v = \psi(r/\varepsilon) e^{im\theta}$, where (r, θ) are polar coordinates on $\mathbb{R}^2$ and $\psi(r/\varepsilon) \to 1$ as $r \to \infty$.

Experimental studies of vortices



Absorption images of a BEC stirred with a laser beam.
From Madison et al., 2000.

Experimental studies of vortex precession

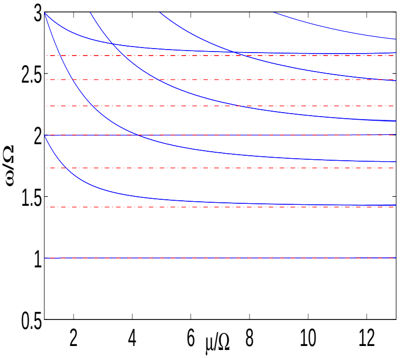


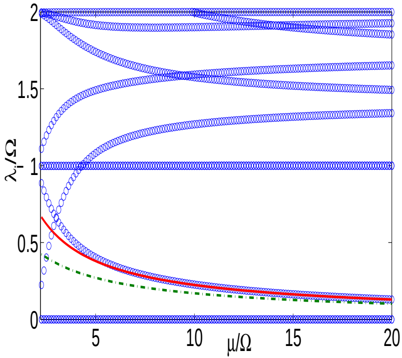
Vortex precession in a trapped two-component BEC.
From Anderson et al., 2000.

Theoretical studies of vortices

- Castin & Dum (1999) - rotating vortices can become local minimizers of energy for larger frequencies
- Fetter & Svidzinsky (2001), Möttönen et al. (2005) - computations of effective energy for vortex configurations
- Aftalion & Du (2001), Ignat & Millot (2006) - variational proofs that a vortex of charge one is a global minimizer for larger frequencies
- Pelinovsky & Kevrekidis (2011) - variational approximations of eigenvalues for single vortices, dipoles and quadrupoles
- Middlecamp et al. (2010), Kollar & Pego (2012) - numerical computations of eigenvalues for vortices, dipoles, and quadrupoles

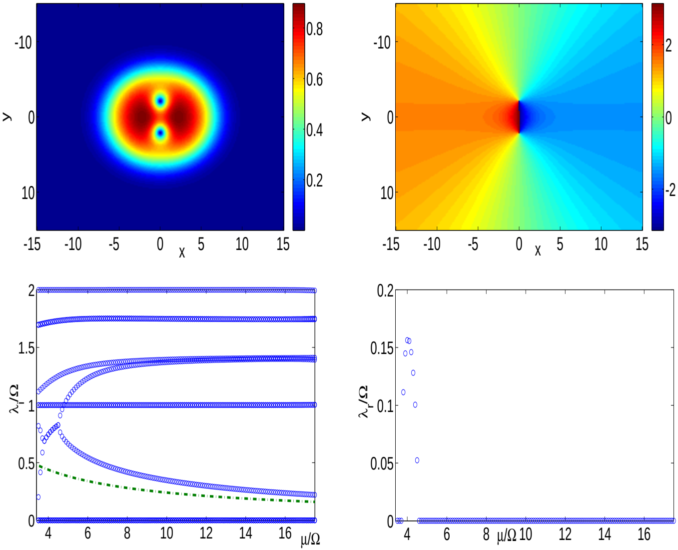
Spectral stability of charge-one vortices





Left: ground state η_ε. Right: charge-one vortex.

Spectral stability of dipole configurations



Variational approximations of eigenvalues

The equivalent Gross–Pitaevskii equation

$$i\,\varepsilon\,\eta_\varepsilon^2 v_t + \varepsilon^2\,\nabla_x\left(\eta_\varepsilon^2 \nabla_x v\right) + \eta_\varepsilon^4 (1 - |v|^2) v = 0,$$

is the Euler–Lagrange equation for the Lagrangian $L(v) = K(v) + \Lambda(v)$ with the kinetic energy

$$K(v) = \frac{i}{2}\,\varepsilon \int_{\mathbb{R}^2} \eta_\varepsilon^2 (v \bar{v}_t - \bar{v} v_t) dx$$

and the potential energy

$$\Lambda(v) = \varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2 |\nabla_x v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \eta_\varepsilon^4 (1 - |v|^2)^2 dx.$$

Substituting a vortex ansatz for $v = \psi(r/\varepsilon) e^{im\theta}$ and computing Euler–Lagrange equations for parameters of the ansatz yield the system of equations that captures qualitative dynamics of a single vortex placed in a harmonic potential.

Free vortex of the defocusing NLS equation

If $\eta_\varepsilon \equiv 1$, a single vortex of charge m of the defocusing NLS equation is given by

$$V_m(x) = \Psi_m(R) e^{im\theta}, \quad R = \frac{r}{\varepsilon}, \quad m \in \mathbb{N},$$

where Ψ_m is a solution of the differential equation

$$\Psi_m'' + R^{-1}\Psi_m' - m^2 R^{-2}\Psi_m + (1 - \Psi_m^2)\Psi_m = 0, \quad R > 0,$$

such that $\Psi_m(0) = 0$, $\Psi_m(R) > 0$ for all $R > 0$, and $\lim_{R\to\infty} \Psi_m(R) = 1$.

The short-range asymptotics is

$$\Psi_m(R) = \alpha_m R^m + \mathcal{O}(R^{m+2}) \quad \text{as} \quad R \to 0.$$

The long-range asymptotics is

$$\Psi_m^2(R) = 1 - \frac{m^2}{R^2} + \mathcal{O}\left(\frac{1}{R^4}\right) \quad \text{as} \quad R \to \infty.$$

Kinetic energy

We can use variables

$$x = x_0 + \varepsilon X, \quad y = y_0 + \varepsilon Y,$$

and write the kinetic energy as

$$K(V_m) = -\dot{x}_0 K_x(V_m) - \dot{y}_0 K_y(V_m),$$

where

$$K_x(V_m) = -m\varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \frac{Y \Psi_m^2}{R^2} dXdY, \quad K_y(V_m) = m\varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \frac{X \Psi_m^2}{R^2} dXdY.$$

Lemma (D.P. & P.Kevrekidis (2011))

For small $\varepsilon > 0$ and small $(x_0, y_0) \in \mathbb{R}^2$, the kinetic energy of a single vortex is represented by

$$K(V_m) = \pi m \varepsilon (x_0 \dot{y}_0 - y_0 \dot{x}_0) \left[1 + \mathcal{O}(\varepsilon) + \mathcal{O}(x_0^2 + y_0^2) \right].$$

Potential energy

We write the potential energy as

$$\Lambda(V_m) = \varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \left[\left(\frac{d\Psi_m}{dR} \right)^2 + \frac{m^2}{R^2} \Psi_m^2 \right] dXdY + \frac{\varepsilon^2}{2} \int_{\mathbb{R}^2} \eta_\varepsilon^4(x)(1 - \Psi_m^2)^2 dXdY.$$

Lemma (D.P. & P.Kevrekidis (2011))

For small $\varepsilon > 0$ and small $(x_0, y_0) \in \mathbb{R}^2$, the potential energy of a single vortex is represented by

$$\Lambda(V_m) - \Lambda(V_m)|_{x_0=y_0=0} = -\pi \, \varepsilon \, m\omega_m(x_0^2 + y_0^2) \left[1 + \mathcal{O}(\varepsilon^{1/3}) + \mathcal{O}(x_0^2 + y_0^2) \right],$$

where ω_m is given by

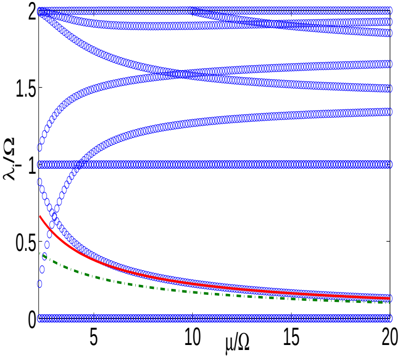
$$\omega_m = \varepsilon \, m \left[2|\log(\varepsilon)| + 1 + \frac{2}{m^2} \int_0^\infty \left[\left(\frac{d\Psi_m}{dR} \right)^2 + \frac{m^2}{R^2} \left(\Psi_m^2 - \frac{R^2}{1+R^2} \right) \right] RdR \right].$$

Eigenfrequencies of the charge-one vortex

Euler–Lagrange equations for the leading part of $L(V_m) = K(V_m) + \Lambda(V_m)$ give

$$-\dot{x}_0 = \omega_m y_0, \quad \dot{y}_0 = \omega_m x_0,$$

where $\omega_m = 2\,\varepsilon\, m|\log(\varepsilon)| + \mathcal{O}(\varepsilon)$. The frequency of vortex precession can be compared with the numerical results plotted here for $\mu = (2\,\varepsilon)^{-1}$ and $\omega = 2\lambda$.



From variational approximations to bifurcation theory

A vortex of charge one has frequency $\omega_1(\varepsilon)$,

$$\omega_1(\varepsilon) = 2\,\varepsilon\,|\log(\varepsilon)| + \mathcal{O}(\varepsilon),$$

which corresponds to its periodic precession around the origin $(0,0) \in \mathbb{R}^2$ with an infinitesimal displacement from the origin.

Consider again the Gross–Pitaevskii equation in the semi-classical form

$$i\,\varepsilon\,u_t + \varepsilon^2\,\nabla_x^2 u + (1 - |x|^2 - |u|^2)u = 0.$$

The static vortex u_ε is a symmetric vortex located at the origin.

Q: Can we find a steadily rotating vortex displaced from the origin at a small but finite distance?

Q: If we can, is this steadily rotating vortex more stable or less stable than the symmetric vortex located at the origin?

Main results I

Theorem (D.P. & P.Kevrekidis (2013))

For every $\epsilon \in \left(0, \frac{1}{4}\right)$, there exists a unique classical solution $u = \psi_\epsilon(r) e^{i\theta}$ for the symmetric vortex of charge one.

For small $|\epsilon - \frac{1}{4}|$, the symmetric vortex is spectrally stable in the sense that all eigenvalues of the spectral stability problem are purely imaginary and semi-simple, except for the double zero eigenvalue.

The symmetric vortex is a saddle point of the Gross–Pitaevskii energy

$$E(u) = \int_{\mathbb{R}^2} \left(\varepsilon^2 |\nabla u|^2 + (|x|^2 - 1)|u|^2 + \frac{1}{2}|u|^4 \right) dx$$

with exactly two eigen-directions for energy decrease (corresponding to eigen-modes of the spectral stability problem with negative Krein signature).

Main results II

Theorem (D.P. & P.Kevrekidis (2013))

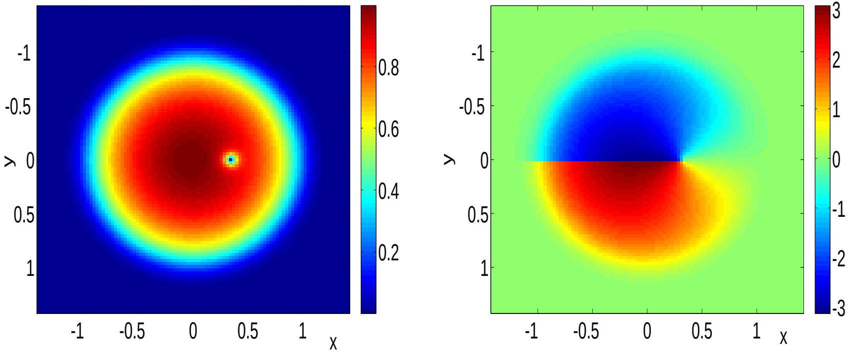
For every $\epsilon \in \left(0, \frac{1}{2}\right)$ with small $|\epsilon - \frac{1}{2}|$, there is a rotational frequency $\omega_0 \in (0, 2)$ such that for every $\omega > \omega_0$ with small $|\omega - \omega_0|$, in addition to the symmetric vortex $u = \psi_\varepsilon(r) e^{i\theta}$, there exists an asymmetric vortex solution $u = u_\varepsilon(x; \alpha)$ of the Gross–Pitaevskii equation with a rotational term.

The center of the asymmetric vortex solution is placed on the circle of radius $|a|$ centered at the origin $(0, 0) \in \mathbb{R}^2$ at an arbitrary angle α. There is $C > 0$ such that

$$|a| \leq C\sqrt{\epsilon(\omega - \omega_0)}.$$

For $\omega > \omega_0$, the symmetric vortex is a local minimizer of the energy $E(u)$, whereas the asymmetric vortex is a saddle point of the energy $E(u)$ with exactly one eigen-direction for energy decrease.

Steady precession of asymmetric charge-one vortices



Spatial contour plots of the amplitude (left) and phase (right) of a rotating charge-one vortex.

Main results III

The energy space of the Gross–Pitaevskii equation

$$X = \left\{ u \in H^1(\mathbb{R}^2) : \ |x|u \in L^2(\mathbb{R}^2) \right\}.$$

Theorem (D.P. & P.Kevrekidis (2013))

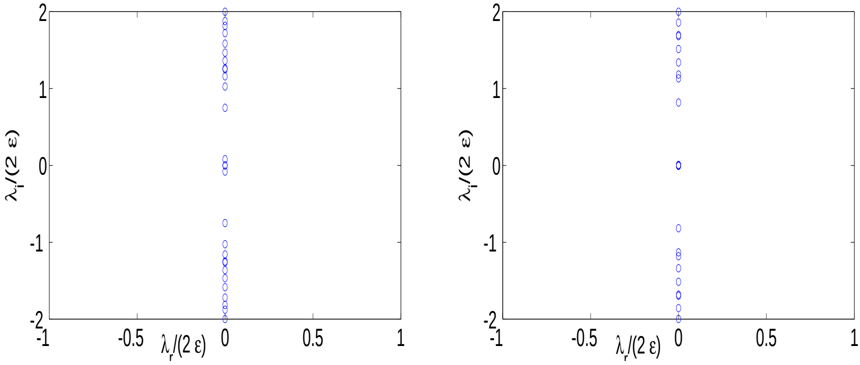
For every $\epsilon \in \left(0, \frac{1}{2}\right)$ and $\omega > \omega_0$ with small $|\epsilon - \frac{1}{2}|$ and $|\omega - \omega_0|$, the symmetric vortex of charge one is orbitally stable in the following sense: for any $\sigma > 0$ there is a $\delta > 0$, such that if $\|u(x,0) - \psi_\varepsilon(r)e^{i\theta}\|_X \leq \delta$, then

$$\inf_{\beta \in \mathbb{R}} \|u(x,t) - e^{i\beta}\psi_\varepsilon(r)e^{i\theta}\|_X \leq \sigma, \quad t \in \mathbb{R}_+,$$

At the same time, the asymmetric vortex is also orbitally stable in the following sense: for any $\sigma > 0$ there is a $\delta > 0$, such that if $\|u(x,0) - u_\varepsilon(x,0)\|_X \leq \delta$, then

$$\inf_{(\alpha,\beta) \in \mathbb{R}^2} \|u(x,t) - e^{i\beta}u_\varepsilon(x;\alpha)\|_X \leq \sigma, \quad t \in \mathbb{R}_+.$$

Spectral stability of rotating charge-one vortices



Left: eigenvalues of the spectral stability problem for the symmetric vortex.
Right: eigenvalues of the spectral stability problem for the asymmetric vortex.

Steadily rotating vortices

In the rotating coordinate frame,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad \omega \in \mathbb{R},$$

the Gross–Pitaevskii equation takes the form,

$$i\,\varepsilon\, u_t + \varepsilon^2 (u_{\xi\xi} + u_{\eta\eta}) + (1 - \xi^2 - \eta^2 - |u|^2) u - i\,\varepsilon\,\omega(\xi u_\eta - \eta u_\xi) = 0.$$

The symmetric vortex of charge one is given by

$$\frac{u(\xi,\eta)}{\sqrt{1+\varepsilon\,\omega}} = \psi_\nu(r) e^{i\theta}, \quad \frac{\sqrt{\xi^2+\eta^2}}{\sqrt{1+\varepsilon\,\omega}} = r,$$

where ψ_ν satisfies the differential equation

$$\nu^2 \left(\frac{d^2\psi_\nu}{dr^2} + \frac{1}{r}\frac{d\psi_\nu}{dr} - \frac{\psi_\nu}{r^2} \right) + (1 - r^2 - \psi_\nu^2)\psi_\nu = 0, \quad \nu = \frac{\varepsilon}{1+\varepsilon\,\omega}.$$

Existence of symmetric vortex

Consider the Schrödinger operator for the quantum harmonic oscillator

$$H(\nu) := -\nu^2(\partial_x^2 + \partial_y^2) + x^2 + y^2 - 1.$$

The spectrum of $H(\nu)$ in $L^2(\mathbb{R}^2)$ is purely discrete:

$$\sigma(H(\nu)) = \left\{\lambda_{n,m}(\nu) = -1 + 2\nu(n+m+1), \quad (n,m) \in \mathbb{N}_0^2\right\},$$

- $\nu = \frac{1}{2}$ - bifurcation of a ground state $\eta_\nu(r)$ $(n = m = 0)$.
- $\nu = \frac{1}{4}$ - bifurcation of a charge-one vortex $\psi_\nu(r)e^{i\theta}$ $(n + m = 1)$.

Lemma

Let $\mu := \frac{1}{16} - \nu^2 > 0$ *and* $\psi_0(r) = re^{-2r^2}$*. Then,*

$$\sup_{r\in\mathbb{R}_+} \left|\psi_\nu(r) - \mu^{1/2}\psi_0(r)\right| \leq C\mu^{3/2}.$$

Energy of the symmetric vortex

Substituting

$$u(x,y) = \psi_\nu(r)e^{i\theta} + U(x,y)$$

to the energy functional $E(u)$, we obtain

$$E(u) - E(\psi_\nu e^{i\theta}) = \langle \mathbf{U}, \mathcal{H}(\nu)\mathbf{U}\rangle_{L^2} + \mathcal{O}(\|\mathbf{U}\|^3_{H^1}), \tag{1}$$

where $\mathbf{U} = [U, \bar{U}]^T$. Using the decomposition in normal modes,

$$U(x,y) = \sum_{m\in\mathbb{Z}} V_m(r)e^{im\theta}, \quad \bar{U}(x,y) = \sum_{m\in\mathbb{Z}} W_m(r)e^{im\theta},$$

we obtain an uncoupled eigenvalue problem for components (V_m, W_{m-2}):

$$H_m(\nu)\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix} = \nu\lambda \begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix}, \quad m \in \mathbb{Z},$$

where

$$H_m(\nu) = \begin{bmatrix} -\nu^2\Delta_m + r^2 - 1 + 2\psi_\nu^2 & \psi_\nu^2 \\ \psi_\nu^2 & -\nu^2\Delta_{m-2} + r^2 - 1 + 2\psi_\nu^2 \end{bmatrix}.$$

Symmetric vortex as a saddle point of energy

Recall

$$H_m(\nu) = \begin{bmatrix} -\nu^2\Delta_m + r^2 - 1 + 2\psi_\nu^2 & \psi_\nu^2 \\ \psi_\nu^2 & -\nu^2\Delta_{m-2} + r^2 - 1 + 2\psi_\nu^2 \end{bmatrix}.$$

and

$$\sigma(-\nu^2\Delta_m + r^2 - 1) = \{\lambda_{n,m}(\nu) = -1 + 2\nu(n+m+1), \quad n \in \mathbb{N}_0\}.$$

Lemma

For $\nu < \frac{1}{4}$ with small $|\nu - \frac{1}{4}|$, there exists exactly one negative eigenvalue $\lambda_0(\nu)$, which has algebraic multiplicity two and is associated to the eigenvectors of $H_2(\nu)$ and $H_0(\nu)$. Moreover, λ_0 is a C^1 function of ν satisfying

$$\lim_{\nu \uparrow \frac{1}{4}} \lambda_0(\nu) = -2.$$

The zero eigenvalue of $H_1(\nu)$ is simple and is associated with the gauge symmetry. All other eigenvalues of $H_m(\nu)$ are strictly positive.

Spectral stability of symmetric vortex

Non-self-adjoint spectral problem:

$$H_m(\nu)\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix} = \nu\lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix}, \quad m \in \mathbb{Z}.$$

Lemma

For $\nu < \frac{1}{4}$ with small $|\nu - \frac{1}{4}|$, the spectral problem admits only real eigenvalues λ of equal algebraic and geometric multiplicities, in addition to the double zero eigenvalue for $m = 1$.

The smallest nonzero eigenvalues are $\lambda = +\omega_0(\nu)$ for $m = 2$ and $\lambda = -\omega_0(\nu)$ for $m = 0$, where $\omega_0(\nu) > 0$ and $\lim_{\nu\uparrow\frac{1}{4}} \omega_0(\nu) = 2$. These eigenvalues are simple and correspond to the eigenvectors $\mathbf{V}_\pm(\nu)$ such that

$$\langle \mathbf{V}_+(\nu), H_2(\nu)\mathbf{V}_+(\nu)\rangle_{L^2_r} = \langle \mathbf{V}_-(\nu), H_0(\nu)\mathbf{V}_-(\nu)\rangle_{L^2_r} < 0.$$

The quadratic form associated with operators $H_m(\nu)$ is strictly positive for the eigenvectors corresponding to any other eigenvalue of the spectral problems.

Two linearizations in the case of rotation

If we substitute $u(\xi,\eta,t) = \psi_\nu(r)e^{i\theta} + U(\xi,\eta,t)$ to the Gross–Pitaevskii equation with the rotation and adopt the decomposition

$$U(\xi,\eta,t) = \sum_{m\in\mathbb{Z}} V^{(m)}(\rho)e^{im\theta}e^{-i\sigma t}, \quad \bar{U}(\xi,\eta,t) = \sum_{m\in\mathbb{Z}} W^{(m)}(\rho)e^{im\theta}e^{-i\sigma t},$$

we obtain the spectral stability problem

$$H_\omega^{(m)} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix} = \varepsilon\,\sigma \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix},$$

where

$$H_\omega^{(m)} = \begin{bmatrix} 1-\rho^2+\varepsilon^2\Delta_m+\varepsilon\,\omega m-2\psi^2 & -\psi^2 \\ -\psi^2 & 1-\rho^2+\varepsilon^2\Delta_{m-2}-\varepsilon\,\omega(m-2)-2\psi^2 \end{bmatrix}$$

On the other hand, linearization of the stationary problem is related to the spectrum of the self-adjoint eigenvalue problem

$$H_\omega^{(m)} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix} = \varepsilon\,\lambda \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix}.$$

Zero eigenvalue of $H_\omega^{(m)}$ signals out a bifurcation of the symmetric vortex.

Two linearizations in the case of rotation

If we substitute $u(\xi,\eta,t) = \psi_\nu(r)e^{i\theta} + U(\xi,\eta,t)$ to the Gross–Pitaevskii equation with the rotation and adopt the decomposition

$$U(\xi,\eta,t) = \sum_{m\in\mathbb{Z}} V^{(m)}(\rho)e^{im\theta}e^{-i\sigma t}, \quad \bar{U}(\xi,\eta,t) = \sum_{m\in\mathbb{Z}} W^{(m)}(\rho)e^{im\theta}e^{-i\sigma t},$$

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$$H_\omega^{(m)} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix} = \varepsilon\,\sigma \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix},$$

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On the other hand, linearization of the stationary problem is related to the spectrum of the self-adjoint eigenvalue problem

$$H_\omega^{(m)} \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix} = \varepsilon\,\lambda \begin{bmatrix} V^{(m)} \\ W^{(m-2)} \end{bmatrix}.$$

Zero eigenvalue of $H_\omega^{(m)}$ signals out a bifurcation of the symmetric vortex.

Transformation of linearizations in the case of rotation

Adopting new variables $\rho = r\sqrt{1+\varepsilon\omega}$ and $\nu = \varepsilon/(1+\varepsilon\omega)$, we transform the spectral stability problem to the form,

$$H_m(\nu)\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix} = \nu(\sigma + \omega(m-1))\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix}, \quad m \in \mathbb{Z},$$

and the self-adjoint eigenvalue problem to the form,

$$H_m(\nu)\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix} = \nu\lambda\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix} + \nu\omega(m-1)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_m \\ W_{m-2} \end{bmatrix},$$

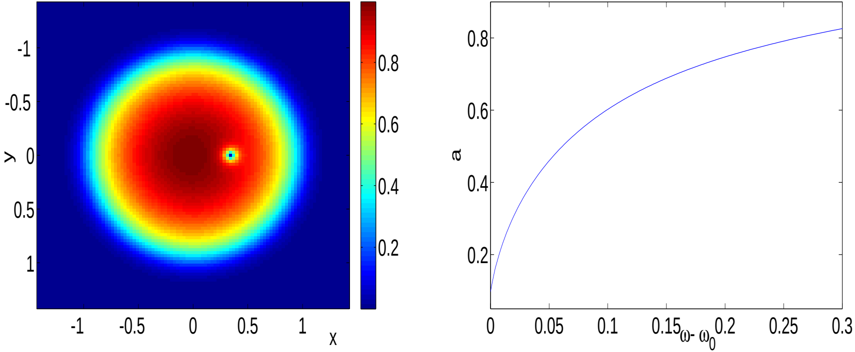
- The symmetric vortex is spectrally stable for

$$\nu \in \left(\nu_0, \frac{1}{4}\right) \quad \Leftrightarrow \quad \epsilon \in \left(\frac{\epsilon_0}{1-\epsilon_0\omega}, \frac{1}{4-\omega}\right).$$

- There exists a bifurcation of the symmetric vortex for $m = 2$ and $\omega = \omega_0(\nu) \in (0,2)$ corresponding to $\epsilon < \frac{1}{2}$. Moreover, if $\lambda(\omega)$ is the eigenvalue such that $\lambda(\omega_0) = 0$, then $\lambda'(\omega_0) > 0$.

Bifurcation of asymmetric vortices

The asymmetric vortex bifurcates for $\omega > \omega_0$ via the supercritical pitchfork bifurcation with radial symmetry. Its center is placed at a point on the circle of radius *a* on the (ξ, η)-plane, where $a \sim \sqrt{\varepsilon(\omega - \omega_0)}$.



Lyapunov–Schmidt reductions I

Root finding problem for $N(u;\omega) : H^2(\mathbb{R}^2) \times \mathbb{R} \to L^2(\mathbb{R}^2)$:

$$N(u;\omega) := -\epsilon^2(u_{\xi\xi} + u_{\eta\eta}) + (\xi^2 + \eta^2 - 1 + |u|^2)u + i\epsilon\omega(\xi u_\eta - \eta u_\xi).$$

The kernel of linearization at the bifurcation point:

$$\mathrm{Ker}(D_u N(\psi_\nu e^{i\theta};\omega_0)) = \mathrm{span}\left\{ \begin{bmatrix} \psi_\nu(r)e^{i\theta} \\ -\psi_\nu(r)e^{-i\theta} \end{bmatrix}, \begin{bmatrix} V_2(r)e^{2i\theta} \\ W_0(r) \end{bmatrix}, \begin{bmatrix} W_0(r) \\ V_2(r)e^{-2i\theta} \end{bmatrix} \right\}.$$

Decomposition

$$u = \psi_\nu(r)e^{i\theta} + aV_2(r)e^{2i\theta} + \bar{a}W_0(r) + U, \quad \omega = \omega_0 + \Omega.$$

Lyapunov–Schmidt reductions II

After a near-identity transformations that eliminates quadratic terms in a, we obtain the normal form equation

$$a\left(\epsilon\Omega\gamma + \beta|a|^2 + \mathcal{O}(|a|^4)\right) = 0,$$

where

$$\gamma = \lambda'(\omega_0)\left(\|V_2\|^2_{L^2_r} + \|W_0\|^2_{L^2_r}\right) > 0$$

and

$$\beta = -\frac{1}{512} + \mathcal{O}(\mu), \quad \mu := \frac{1}{16} - \nu^2 > 0.$$

For small μ, we have the supercritical pitchfork bifurcation with radial symmetry:

$$|a|^2 = 32\epsilon(\omega - \omega_0) + \mathcal{O}((\omega-\omega_0)^2, \mu),$$

and $\alpha = \arg(a)$ is an arbitrary angle in the (ξ,η)-plane for the vortex core on the circle of radius $|a|$.

Orbital stability of vortices

Symmetric vortex of charge one $u = \psi_\nu(r)e^{i\theta}$ is a local minimizer of energy $E(u)$ in space X for $\omega > \omega_0$. Therefore, it is orbitally stable in the sense: for any $\sigma > 0$ there is a $\delta > 0$, such that if $\|u(x,0) - \psi_\nu(r)e^{i\theta}\|_X \leq \delta$, then

$$\inf_{\beta \in \mathbb{R}} \|u(x,t) - e^{i\beta}\psi_\nu(r)e^{i\theta}\|_X \leq \sigma, \quad t \in \mathbb{R}_+,$$

The new asymmetric vortex $u = u_\varepsilon(x;\alpha)$ is a saddle point of energy $E(u)$ in space X. The linearization operator $D_u N(u_\varepsilon;\omega)$ has exactly one negative eigenvalue and the two-dimensional kernel:

$$\mathrm{Ker}(D_u N(u_\varepsilon;\omega)) = \mathrm{span}\left\{ \begin{bmatrix} u_\varepsilon \\ -\bar{u}_\varepsilon \end{bmatrix}, \begin{bmatrix} \partial_\alpha u_\varepsilon \\ \partial_\alpha \bar{u}_\varepsilon \end{bmatrix} \right\}.$$

We show that this vortex is also orbitally stable in the sense: for any $\sigma > 0$ there is a $\delta > 0$, such that if $\|u(x,0) - u_\varepsilon(x,0)\|_X \leq \delta$, then

$$\inf_{(\alpha,\beta) \in \mathbb{R}^2} \|u(x,t) - e^{i\beta}u_\varepsilon(x;\alpha)\|_X \leq \sigma, \quad t \in \mathbb{R}_+.$$

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Stability of the asymmetric vortex

We need to prove that the linearization operator $D_u N(u_\varepsilon;\omega)$ is non-negative in the constrained space

$$L^2_c(\mathbb{R}^2) = \left\{ U \in L^2(\mathbb{R}^2) : \quad \langle \mathbf{V}, \sigma_3 \mathbf{U} \rangle := \int_{\mathbb{R}^2} \left(\bar{V} U - \bar{W} \bar{U} \right) dx = 0, \right.$$
$$\left. \text{for every } \; \mathbf{V} = \left[\begin{array}{c} V \\ W \end{array} \right] \in \mathrm{Ker}(D_u N(u_\varepsilon;\omega)) \right\},$$

where $\sigma_3 = \mathrm{diag}(1,-1)$ respects the symplectic structure of the GP equation.

This result is equivalent to the fact that the matrix of symplectic projections

$$\left[\begin{array}{cc} \langle \mathbf{V}_g, \sigma_3 \tilde{\mathbf{V}}_g \rangle & \langle \mathbf{V}_r, \sigma_3 \tilde{\mathbf{V}}_g \rangle \\ \langle \mathbf{V}_g, \sigma_3 \tilde{\mathbf{V}}_r \rangle & \langle \mathbf{V}_r, \sigma_3 \tilde{\mathbf{V}}_r \rangle \end{array} \right]$$

has exactly one negative eigenvalue, where $\tilde{\mathbf{V}}_g$ and $\tilde{\mathbf{V}}_r$ are generalized eigenvectors of the generalized kernel of $D_u N(u_\varepsilon;\omega)$. When $\nu \to \frac{1}{4}$, this is confirmed by the explicit computations.

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Conclusion and open questions

We have described the local bifurcation results for the birth of steadily rotating asymmetric vortices of charge one in the Gross-Pitaevskii equation with a symmetric harmonic potential.

For supercritical rotational frequency, symmetric vortices of charge one are local minimizers of energy and asymmetric vortices of charge one are saddle points of the energy. Nevertheless, both vortices are orbitally stable with respect to the time-dependent perturbations.

Open questions:

- Can we extend these results to the entire existence interval $(0, \frac{1}{4})$ (in terms of parameter ν)?
- Can we prove these results in the Thomas–Fermi limit $\nu \to 0$?
- Can we consider other local minimizers of energy given by two, three, and many vortices of charge one?

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