Global existence and wave breaking in Burgers-type equations with low-frequency dispersion

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Ostrovsky equation for rotating fluid

The **Ostrovsky equation** is a model for small-amplitude long waves in a rotating fluid of a finite depth [Ostrovsky, 1978]:

$$(u_t + uu_x - \beta u_{xxx})_x = \gamma u,$$

where β and γ are real coefficients.

When $\beta = 0$ and $\gamma = 1$, the Ostrovsky equation is

$$(u_t + uu_x)_x = u,$$

and is known under the names of

- the short-wave equation [Hunter, 1990];
- Ostrovsky–Hunter equation [Boyd, 2005];
- reduced Ostrovsky equation [Stepanyants, 2006];
- the Vakhnenko equation [Vakhnenko & Parkes, 2002].

Short-pulse equation

The **short-pulse equation** is a model for propagation of ultra-short pulses with few cycles on the pulse scale [Schäfer, Wayne 2004]:

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx},$$

where all coefficients are normalized thanks to the scaling invariance.

The short-pulse equation

- replaces the nonlinear Schrödinger equation for short wave packets
- features exact solutions for modulated pulses
- enjoys inverse scattering and an infinite set of conserved quantities

Well-posedness results

- T. Schafer and C.E. Wayne (2004) proved local existence in $H^2(\mathbb{R})$.
- A. Stefanov *et al.* (2010) considered a family of the generalized short-pulse equations
$$u_{xt} = u + (u^p)_{xx}$$
and proved scattering to zero for *small* initial data if $p \geq 4$.
- Y. Liu *et al.* (2009,2010) proved global existence for *small* initial data and wave breaking for *large* initial data if $p = 3$.
- Y. Liu *et al.* (2010) proved wave breaking for sufficiently *large* initial data if $p = 2$ but found no proof of global existence for *small* initial data.
- T. Johnson *et al.* (2012) suggested a sharp criterion that distinguished between global existence and wave breaking for $p = 2$.
- R. Grimshaw, D.P. (2014) proved global existence for *small* initial data.

Integrability of the short-pulse equation

The short-pulse equation is

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \quad x \in \mathbb{R}, \quad t \in [0, T].$$

Integrability of the short-pulse equation

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$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \quad x \in \mathbb{R}, \quad t \in [0, T].$$

Let $x = x(y, t)$ satisfy

$$\begin{cases} x_y = \cos w, \\ x_t = -\frac{1}{2} w_t^2. \end{cases}$$

If $w = w(y, t)$ satisfies the sine–Gordon equation in characteristic coordinates [A. Sakovich, S. Sakovich (2006)]

$$w_{yt} = \sin(w), \quad y \in \mathbb{R}, \quad t \in [0, T],$$

then $u(x, t) = w_t(y(x, t), t)$ solves the short-pulse equation.

The map $\mathbb{R} \ni y \to x \in \mathbb{R}$ is invertible for $t \in [0, T]$, if

$$\cos(w) > 0 \quad \text{or} \quad \|w\|_{L^\infty} < \frac{\pi}{2}.$$

A kink of the sine–Gordon equation gives a *loop solution* of the short-pulse equation:

$$\begin{cases} u = 2\,\mathrm{sech}(y+t), \\ x = y - 2\tanh(y+t). \end{cases}$$

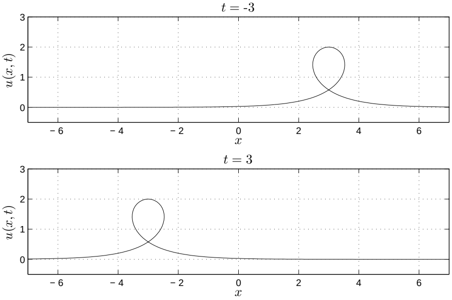

Figure : The loop solution $u(x,t)$ to the short-pulse equation

Solutions of the short-pulse equation

A breather of the sine–Gordon equation gives a *pulse solution* of the short-pulse equation:

$$\begin{cases} u(y,t) = 4mn\dfrac{m\sin\psi\sinh\phi + n\cos\psi\cosh\phi}{m^2\sin^2\psi + n^2\cosh^2\phi} = u\left(y-\frac{\pi}{m}, t+\frac{\pi}{m}\right), \\ x(y,t) = y + 2mn\dfrac{m\sin 2\psi - n\sinh 2\phi}{m^2\sin^2\psi + n^2\cosh^2\phi} = x\left(y-\frac{\pi}{m}, t+\frac{\pi}{m}\right) + \frac{\pi}{m}, \end{cases}$$

where

$$\phi = m(y+t), \quad \psi = n(y-t), \quad n = \sqrt{1-m^2},$$

and $m \in \mathbb{R}$ is a free parameter.

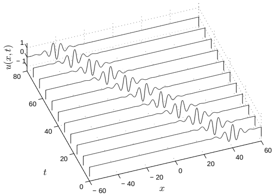

Figure : The pulse solution to the short-pulse equation with $m = 0.25$

Local well-posedness of the short-pulse equation

Theorem (Schäfer & Wayne, 2004)

Let $u_0 \in H^2$. *There exists a maximal existence time* $T = T(u_0) > 0$ *and a unique solution to the short-pulse equation*

$$u(t) \in C([0,T), H^2) \cap C^1([0,T), H^1)$$

that satisfies $u(0) = u_0$ *and depends continuously on* u_0.

Remarks:

- The proof can be extended to any $s > \frac{3}{2}$ (Stefanov *et al*, 2010).
- There is a constraint on solutions of the short-pulse equation

$$\int_{\mathbb{R}} u(x,t)dx = 0, \quad t > 0.$$

A better space is $H^s \cap \dot{H}^{-1}$ for $s > \frac{3}{2}$.

Conserved quantities of the short-pulse equation

A bi-infinite hierarchy of conserved quantities of the short-pulse equation was found in Brunelli [J.Math.Phys. **46**, 123507 (2005)]:

$$\begin{aligned}
&\cdots \\
E_{-1} &= \int_{\mathbb{R}} \left(\frac{1}{24} u^4 - \frac{1}{2} (\partial_x^{-1} u)^2 \right) dx, \\
E_0 &= \int_{\mathbb{R}} u^2 dx, \\
E_1 &= \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} dx, \\
E_2 &= \int_{\mathbb{R}} \frac{u_{xx}^2}{(1 + u_x^2)^{5/2}} dx, \\
&\cdots
\end{aligned}$$

Conserved quantities E_{-1}, E_0, E_1, E_2 are defined in the energy space $H^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$.

Theorem (P. & Sakovich, 2010)

Let $u_0 \in H^2$ *such that* $\|u_0'\|_{L^2}^2 + \|u_0''\|_{L^2}^2 < 1$. *Then the short-pulse equation admits a unique solution* $u(t) \in C(\mathbb{R}, H^2)$ *with* $u(0) = u_0$.

The constant values of E_0, E_1 and E_2 are bounded by $\|u_0\|_{H^2}$ as follows:

$$\begin{aligned} E_0 &= \int_{\mathbb{R}} u^2 dx = \|u_0\|_{L^2}^2, \\ E_1 &= \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} dx \leq \frac{1}{2} \|u_0'\|_{L^2}^2, \\ E_2 &= \int_{\mathbb{R}} \frac{u_{xx}^2}{(1 + u_x^2)^{5/2}} dx \leq \|u_0''\|_{L^2}^2. \end{aligned}$$

so that $2E_1 + E_2 < 1$.

The local existence time $T > 0$ is inverse proportional to the norm $\|u_0\|_{H^2}$ of the initial data u_0. To extend T to ∞, we need to control the norm $\|u(t)\|_{H^2}$ by a T-independent constant on $[0, T]$.

Sketch of the proof

- Let $q(x,t) = \frac{u_x}{\sqrt{1+u_x^2}}$. Then, we obtain

$$\begin{aligned} \|q\|_{L^2}^2 &\leq \int_{\mathbb{R}} \frac{u_x^2}{1+\sqrt{1+u_x^2}} \frac{1+\sqrt{1+u_x^2}}{1+u_x^2} dx \leq 2E_1, \\ \|\partial_x q\|_{L^2}^2 &\leq \int_{\mathbb{R}} \sqrt{1+u_x^2} \left[\partial_x \frac{u_x}{\sqrt{1+u_x^2}} \right]^2 dx = E_2, \end{aligned}$$

  hence, $\|q(t)\|_{H^1} \leq \sqrt{2E_1 + E_2} < 1$, $t \in [0,T]$.

- Thanks to Sobolev's embedding $\|q\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|q\|_{H^1} < 1$, the inverse transformation $u_x = \frac{q}{\sqrt{1-q^2}}$ satisfies the bound

$$\|u_x\|_{H^1} \leq \frac{\|q\|_{H^1}}{\sqrt{1 - \|q\|_{H^1}^2}},$$

  or equivalently

$$\|u(t)\|_{H^2} \leq \left(E_0 + \frac{2E_1 + E_2}{1 - (2E_1 + E_2)} \right)^{1/2}, \quad t \in [0,T].$$

Corollary

Let $u_0 \in H^2$ *such that* $2\sqrt{2E_1E_2} < 1$. *Then the short-pulse equation admits a unique solution* $u(t) \in C(\mathbb{R}, H^2)$ *with* $u(0) = u_0$.

Let $\alpha > 0$ be arbitrary. If $u(x,t)$ is a solution of the short-pulse equation, then $\tilde{u}(\tilde{x},\tilde{t})$ is also a solution of the same equation with

$$\tilde{x} = \alpha x, \quad \tilde{t} = \alpha^{-1} t, \quad \tilde{u}(\tilde{x},\tilde{t}) = \alpha u(x,t).$$

The scaling invariance yields transformation $\tilde{E}_1 = \alpha E_1$ and $\tilde{E}_2 = \alpha^{-1} E_2$. For a given $u_0 \in H^2$, a family of initial data $\tilde{u}_0 \in H^2$ satisfies

$$\phi(\alpha) = 2\tilde{E}_1 + \tilde{E}_2 = 2\alpha E_1 + \alpha^{-1} E_2 \geq 2\sqrt{2E_1E_2}, \quad \forall \alpha > 0.$$

If $2\sqrt{2E_1E_2} < 1$, there exists α such that $\tilde{u}$ is defined for any $\tilde{t} \in \mathbb{R}$.

Criterion for wave breaking

Consider the Cauchy problem for the inviscid Burgers equation

$$\begin{cases} u_t = \frac{1}{2}u^2 u_x, \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{R}, \quad t \geq 0.$$

The Cauchy problem can be solved by the method of characteristics. The finite-time blow-up occurs for any $u_0(x) \in C^1(\mathbb{R})$ if there is a point $x_0 \in \mathbb{R}$ such that $u_0(x_0)u_0'(x_0) > 0$. The blow-up time is

$$T = \inf_{\xi \in \mathbb{R}} \left\{ \frac{1}{u_0(\xi)u_0'(\xi)} : \quad u_0(\xi)u_0'(\xi) > 0 \right\}.$$

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$$T = \inf_{\xi \in \mathbb{R}} \left\{ \frac{1}{u_0(\xi)u_0'(\xi)} : \quad u_0(\xi)u_0'(\xi) > 0 \right\}.$$

Lemma

Let $u_0 \in H^2(\mathbb{R})$ and $u(t)$ be a local solution of the Cauchy problem for the short-pulse equation. The solution blows up in a finite time $T < \infty$ in the sense $\lim_{t \uparrow T} \|u(\cdot,t)\|_{H^2} = \infty$ if and only if

$$\lim_{t \uparrow T} \sup_{x \in \mathbb{R}} u(x,t)u_x(x,t) = +\infty.$$

Short-pulse equation in a periodic domain

The short-pulse equation on the unit circle $\mathbb{S}$ is given by

$$\left\{ \begin{array}{l} u_t = \frac{1}{2} u^2 u_x + \partial_x^{-1} u, \\ u(x,0) = u_0(x), \end{array} \right. \qquad x \in \mathbb{S}, \quad t \geq 0,$$

where $\partial_x^{-1} u$ is the mean-zero anti-derivative,

$$\partial_x^{-1} u = \int_0^x u(x',t)dx' - \int_{\mathbb{S}} \int_0^x u(x',t)dx'dx.$$

- The assumption $\int_{\mathbb{S}} u_0(x)dx = 0$ is necessary for existence.
- The following quantities are constant as long as the solution exists:

$$E_0 = \int_{\mathbb{S}} u^2 dx, \quad E_1 = \int_{\mathbb{S}} \sqrt{1 + u_x^2} dx$$

Method of characteristics

Let $\xi \in \mathbb{S}$, $t \in [0, T)$, and denote

$$x = X(\xi, t), \quad u(x,t) = U(\xi, t), \quad \partial_x^{-1} u(x,t) = G(\xi, t).$$

At characteristics $x = X(\xi, t)$, we obtain

$$\left\{ \begin{array}{l} \dot{X}(t) = -\frac{1}{2} U^2, \\ X(0) = \xi, \end{array} \right. \qquad \left\{ \begin{array}{l} \dot{U}(t) = G, \\ U(0) = u_0(\xi), \end{array} \right.$$

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Both U and G are bounded on the smooth solutions:

$$|u(x,t)| \leq \int_{\mathbb{S}} |u_x(x,t)| dx \leq E_1$$

and

$$|\partial_x^{-1} u(x,t)| \leq \int_{\mathbb{S}} |u(x,t)| dx \leq \sqrt{E_0}.$$

Sufficient condition for wave breaking

Theorem (Liu, P. & Sakovich, 2009)

Let $u_0 \in H^2(\mathbb{S})$ *and* $\int_{\mathbb{S}} u_0(x)\,dx = 0$. *Assume that there exists* $x_0 \in \mathbb{R}$ *such that* $u_0(x_0)u_0'(x_0) > 0$ *and*

$$\text{either} \qquad |u_0'(x_0)| > \left(\frac{E_1^2}{4E_0^{1/2}}\right)^{1/3},$$

$$|u_0(x_0)||u_0'(x_0)|^2 > E_1 + \left(2E_0^{1/2}|u_0'(x_0)|^3 - \frac{1}{2}E_1^2\right)^{1/2},$$

$$\text{or} \qquad |u_0'(x_0)| \le \left(\frac{E_1^2}{4E_0^{1/2}}\right)^{1/3}, \quad |u_0(x_0)||u_0'(x_0)|^2 > E_1.$$

Then there exists a finite time $T \in (0,\infty)$ *such that the solution* $u(t) \in C([0,T), H^2(\mathbb{S}))$ *blows up with the property*

$$\lim_{t\uparrow T} \sup_{x\in\mathbb{S}} u(x,t)u_x(x,t) = +\infty, \quad \textit{while} \quad \lim_{t\uparrow T} \|u(\cdot,t)\|_{L^\infty} \le E_1.$$

Sketch of the proof

Let $V(\xi, t) = u_x(X(\xi, t), t)$ and $W(\xi, t) = U(\xi, t)V(\xi, t)$. Then

$$\begin{cases} \dot{V} & = & VW + U, \\ \dot{W} & = & W^2 + VG + U^2. \end{cases}$$

Under the conditions of the theorem, there exists $\xi_0 \in \mathbb{S}$ such that $V(\xi_0, t)$ and $W(\xi_0, t)$ satisfy the apriori estimates

$$\begin{cases} \dot{V} & \geq & VW - E_1, \\ \dot{W} & \geq & W^2 - V\sqrt{E_0}. \end{cases}$$

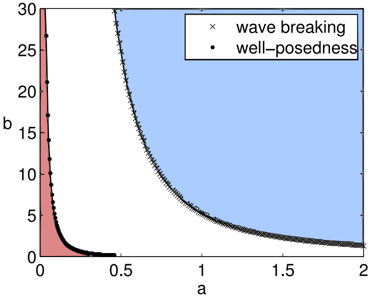
By comparison theorem, $V(\xi_0, t) \geq \mathbf{V}(\xi_0, t)$ and $W(\xi_0, t) \geq \mathbf{W}(\xi_0, t)$, where the lower solution $(\mathbf{V}, \mathbf{W})$ diverges to infinity in a finite time.

Criteria of well-posedness and wave breaking

Consider Gaussian initial data

$$u_0(x) = a(1 - 2bx^2)e^{-bx^2}, \quad x \in \mathbb{R},$$

where (a, b) are arbitrary and $\int_{\mathbb{R}} u_0(x)dx = 0$ is satisfied.



Global solutions exist in the red region and wave breaking occurs in the blue region.

Using the pseudospectral method, we solve

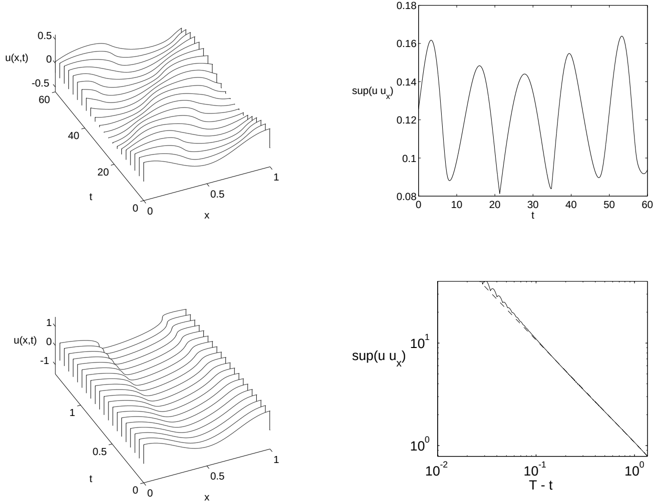
$$\frac{\partial}{\partial t}\hat{u}_k = -\frac{i}{k}\hat{u}_k + \frac{ik}{6}\mathcal{F}\left[\left(\mathcal{F}^{-1}\hat{u}\right)^3\right]_k, \quad k \neq 0, \quad t > 0.$$

Consider the 1-periodic initial data

$$u_0(x) = a\cos(2\pi x)$$

- Criterion for wave breaking: $a > 1.053$.
- Criterion for global solutions: $a < 0.0354$.

Evolution of the cosine initial data



Solution surface $u(x,t)$ (left) and the supremum norm $W(t)$ (right) for $a = 0.2$ (top) and $a = 0.5$ (bottom).

Local well-posedness of the reduced Ostrovsky equation

Theorem (Stefanov *et al.*, 2010)

Let $u_0 \in H^s$, $s > \frac{3}{2}$. There exists a maximal existence time $T = T(u_0) > 0$ and a unique solution to the reduced Ostrovsky equation $(u_t + uu_x)_x = u$,

$$u(t) \in C([0,T), H^s) \cap C^1([0,T), H^{s-1}),$$

that satisfies $u(0) = u_0$ and depends continuously on u_0.

Integrability is based on the reduction to the Vakhnenko equation (Vakhnenko, 1992), which is a sort of Hirota–Satsuma equation with a reversed role of space-time variables. The conserved quantities are:

$$\begin{aligned} & \cdots \\ E_{-1} &= \int_{\mathbb{R}} \left(\frac{1}{3} u^3 + (\partial_x^{-1} u)^2 \right) dx, \\ E_0 &= \int_{\mathbb{R}} u^2 dx. \end{aligned}$$

Conserved quantities are not helpful to control solution in H^s, $s > \frac{3}{2}$.

Theorem (Grimshaw & P., 2014)

Let $u_0 \in H^3$ *such that* $1 - 3u_0''(x) > 0$ *for all* x. *Then the reduced Ostrovsky equation admits a unique solution* $u(t) \in C(\mathbb{R}, H^3)$ *with* $u(0) = u_0$.

This result is based on the number of preliminary works:

- Hone & Wang (2003) obtained Lax pair
$$\begin{cases} 3\lambda\psi_{xxx} + (1 - 3u_{xx})\psi = 0, \\ \psi_t + \lambda\psi_{xx} + u\psi_x - u_x\psi = 0, \end{cases}$$
- Kraenkel *et al.* (2011) showed equivalence with the Bullough–Dodd (Tzitzeica) equation
$$\frac{\partial^2 V}{\partial t \partial z} = e^{-2V} - e^{V}.$$
- Grimshaw *et al.* (2013) suggested the relevance of $1 - 3u_0''(x)$ from asymptotic and numerical analysis.

Conserved quantities for the reduced Ostrovsky equation

Brunelli & Sakovich (2013) found bi-infinite sequence of conserved quantities for the reduced Ostrovsky equation:

$$
\begin{aligned}
& \dots \\
E_{-1} &= \int_{\mathbb{R}} \left(\frac{1}{3} u^3 + (\partial_x^{-1} u)^2 \right) dx, \\
E_0 &= \int_{\mathbb{R}} u^2 dx \\
E_1 &= \int_{\mathbb{R}} \left[(1 - 3u_{xx})^{1/3} - 1 \right] dx, \\
E_2 &= \int_{\mathbb{R}} \frac{(u_{xxx})^2}{(1 - 3u_{xx})^{7/3}} dx \\
& \dots
\end{aligned}
$$

However, the quantity $1 - 3u_{xx}$ needs to be controlled over the time span.

Characteristic variables for the reduced Ostrovsky equation

Starting with the reduced Ostrovsky equation

$$(u_t + uu_x)_x = u, \quad x \in \mathbb{R}, \quad t \in [0, T].$$

Let $x = x(y, t)$ satisfy $x = y + \int_0^t U(y, t')dt'$ with $u(x, t) = U(y, t)$. The transformation $y \to x$ is invertible if

$$\phi(y, t) = 1 + \int_0^t U_y(y, t')dt' \neq 0.$$

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$$\phi(y,t) = 1 + \int_0^t U_y(y,t')dt' \neq 0.$$

Let us introduce $f(x,t) = (1 - 3u_{xx})^{1/3} = F(y,t)$. Then,

$$f_t + (uf)_x = 0 \quad (F\phi)_t = 0.$$

so that $F(y,t)\phi(y,t) = F_0(y)$.

The reduced Ostrovsky equation is equivalent to the evolution equation

$$\frac{\partial^2}{\partial t \partial y} \log(F) = \frac{1}{3} F_0(y)(F^2 - F^{-1}).$$

Sketch of the proof

- If $1 - 3u_0''(x) > 0$ for all $x \in \mathbb{R}$, then $F_0(y) > 0$. We introduce

$$z := -\frac{1}{3}\int_0^y F_0(y')dy', \quad F(y,t) := e^{-V(z,t)},$$

and obtain the Tzitzéica equation

$$\frac{\partial^2 V}{\partial t \partial z} = e^{-2V} - e^{V}.$$

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and obtain the Tzitzéica equation

$$\frac{\partial^2 V}{\partial t \partial z} = e^{-2V} - e^{V}.$$

- There exists a unique local solution of the Tzitzéica equation in class $V \in C([0,T], H^1(\mathbb{R}))$ for some $T > 0$ such that $V(z,0) = V_0(z)$:

$$V(z,t) = -\frac{1}{3}\log\left(1 - 3u_{xx}(x,t)\right).$$

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$$V(z,t) = -\frac{1}{3}\log\left(1 - 3u_{xx}(x,t)\right).$$

- The solution is extended globally in class $V \in C(\mathbb{R}, H^1(\mathbb{R}))$ thanks to the conserved quantities

$$Q_1 = \int_{\mathbb{R}} \left(2e^{V} + e^{-2V} - 3\right) dz, \quad Q_2 = \int_{\mathbb{R}} \left(\frac{\partial V}{\partial z}\right)^2 dz.$$

- This yields a global solution to the reduced Ostrovsky equation in class $u \in C(\mathbb{R}, H^3(\mathbb{R}))$.

The reduced Ostrovsky equation

Consider the Cauchy problem on a circle $\mathbb{S}$ of unit length:

$$\begin{cases} u_t + uu_x = \partial_x^{-1} u, & t > 0, \\ u(0,x) = u_0(x). \end{cases}$$

The inviscid Burgers equation $u_t + uu_x = 0$ develops wave breaking in a finite time for any initial data $u(0,x) = u_0(x)$ if $u_0(x) \in C^1$ and there is a point x_0 such that $u_0'(x_0) < 0$. The blow-up time is computed by the method of characteristics:

$$T = \inf_{\xi} \left\{ \frac{1}{|u_0'(\xi)|} : \quad u_0'(\xi) < 0 \right\}.$$

Lemma

Let $u_0 \in H^2(\mathbb{S})$ and $u(t)$ be a local solution of the Cauchy problem for the reduced Ostrovsky equation. The solution blows up in a finite time $T < \infty$ in the sense $\lim_{t \uparrow T} \|u(\cdot,t)\|_{H^2} = \infty$ if and only if

$$\liminf_{t \uparrow T} \inf_x u_x(t,x) = -\infty, \quad \textit{while} \quad \limsup_{t \uparrow T} \sup_x |u(t,x)| < \infty.$$

Sufficient results for wave breaking

Theorem (Hunter, 1990)

Let $u_0(x) \in C^1(\mathbb{S})$*, where* $\mathbb{S}$ *is a circle of unit length, and define*

$$\inf_{x\in\mathbb{S}} u_0'(x) = -m \quad \text{and} \quad \sup_{x\in\mathbb{S}} |u_0(x)| = M.$$

If $m^3 > 4M(4+m)$*, a smooth solution* $u(t,x)$ *breaks down at a finite time.*

Theorem (Liu, P. & Sakovich, 2010)

Assume that $u_0(x) \in H^s(\mathbb{S})$*,* $s > \frac{3}{2}$ *and* $\int_{\mathbb{S}} u_0(x)\,dx = 0$*. If either*

$$\int_{\mathbb{S}} \left(u_0'(x)\right)^3 \, dx < -\left(\frac{3}{2}\|u_0\|_{L^2}\right)^{3/2}, \tag{1}$$

or there is a $x_0 \in \mathbb{S}$ *such that*

$$u_0'(x_0) < -1\left(\|u_0\|_{L^\infty} + T_1\|u_0\|_{L^2}\right)^{\frac{1}{2}}, \tag{2}$$

then the solution $u(t,x)$ *of the Cauchy problem blows up in a finite time.*

Proof of the sufficient condition (1)

Direct computation gives

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx &= 3\int_{\mathbb{S}} u_x^2\left(-u_x^2 - uu_{xx} + u\right)\,dx\\ &= -2\int_{\mathbb{S}} u_x^4\,dx + 3\int_{\mathbb{S}} uu_x^2\,dx\\ &\leq -2\|u_x\|_{L^4}^4 + 3\|u\|_{L^2}\|u_x\|_{L^4}^2.\end{aligned}$$

By Hölder's inequality, we have

$$|V(t)| \leq \|u_x\|_{L^3}^3 \leq \|u_x\|_{L^4}^3, \quad V(t) = \int_{\mathbb{S}} u_x^3(t,x)\,dx < 0.$$

Let $Q_0 = \|u\|_{L^2}^2 = \|u_0\|_{L^2}^2$ and $V(0) < -\left(\frac{3}{2}Q_0\right)^{\frac{3}{2}}$. Then,

$$\frac{dV}{dt} \leq -2\left(|V|^{\frac{2}{3}} - \frac{3Q_0}{4}\right)^2 + \frac{9Q_0^2}{8},$$

There is $T < \infty$ such that $V(t) \to -\infty$ as $t \uparrow T$.

Proof of the sufficient condition (2)

Let $\xi \in \mathbb{S}$, $t \in [0, T)$, and denote

$$x = X(\xi, t), \quad u(x,t) = U(\xi, t), \quad \partial_x^{-1} u(x,t) = G(\xi, t).$$

At characteristics $x = X(\xi, t)$, we obtain

$$\begin{cases} \dot{X}(t) = U, \\ X(0) = \xi, \end{cases} \qquad \begin{cases} \dot{U}(t) = G, \\ U(0) = u_0(\xi), \end{cases}$$

Let $V(\xi, t) = u_x(t, X(\xi, t))$. Then

$$\dot{V} = -V^2 + U \quad \Rightarrow \quad \dot{V} \leq -V^2 + (\|u_0\|_{L^\infty} + t\|u_0\|_{L^2})$$

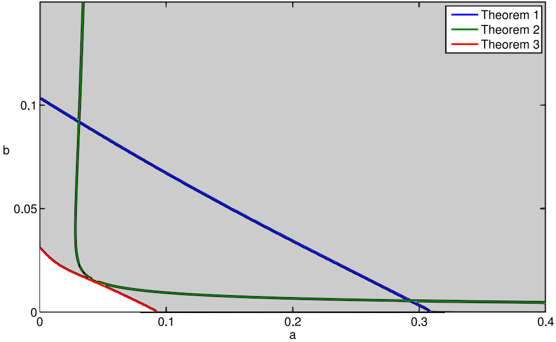
There is $T < \infty$ such that $V(t) \to -\infty$ as $t \uparrow T$.

Using the pseudospectral method, we solve

$$\frac{\partial}{\partial t}\hat{u}_k = -\frac{i}{k}\hat{u}_k - \frac{ik}{2}\mathcal{F}\left[\left(\mathcal{F}^{-1}\hat{u}\right)^2\right]_k, \quad k \neq 0, \quad t > 0.$$

Consider the 1-periodic initial data

$$u_0(x) = a\cos(2\pi x) + b\sin(4\pi x),$$



Evolution of the cosine initial data

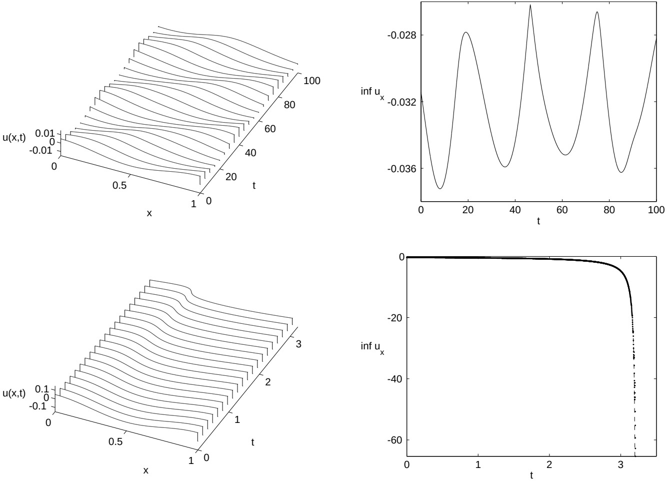


Figure : Solution surface $u(t,x)$ (left) and $\inf_{x\in\mathbb{S}} u_x(t,x)$ versus t (right) for $a = 0.005$, $b = 0$ (top) and $a = 0.05$, $b = 0$ (bottom). $C \approx -1.009$ and $B \approx 3.213$.

Summary

For both the short-pulse and reduced Ostrovsky equations, we have ...

- ... found sufficient conditions for global well-posedness for small data.
- ... found sufficient conditions for wave breaking for large initial data.
- ... illustrated both global existence and wave breaking numerically.

For the reduced Ostrovsky equation, there is a sharp criterion on the initial data for the global solutions to exist.

It is not clear if a similar sharp criterion on the initial data exists for the short-pulse equation.