Instability of peaked waves in hydrodynamical models

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SIAM Conference on Nonlinear Waves and Coherent Structures
Baltimore, USA, June 24-27 2024

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In collaboration with
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Related to MS30-MS36: Peakons: Existence, Stability, and Beyond
and MS6-MS12: Stability of Traveling Waves
and MS14-MS21: Exploring Nonlinear Waves and Singularities
and MS35-MS43: Water Waves

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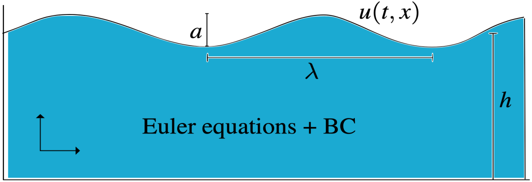
Background

Traveling waves for the irrotational motion of an incompressible fluid:



Background

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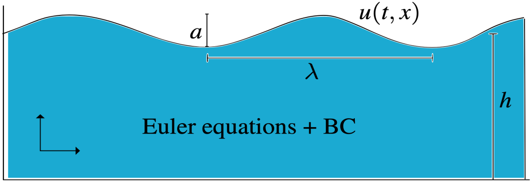
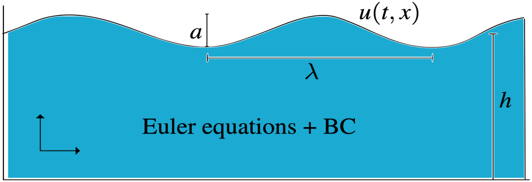


How do the waves break?

Background

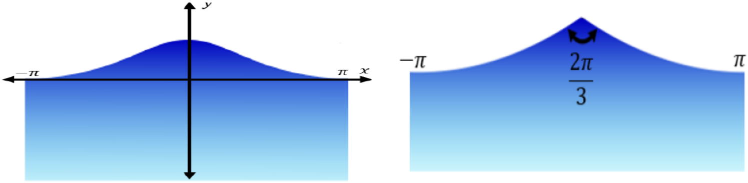
Traveling waves for the irrotational motion of an incompressible fluid:



This has a long history starting with Sir George Stokes (1819-1903)

Background

In 1880 Stokes suggested existence of the peaked wave in the family of smooth traveling waves:



Existence of the peaked wave was proven by Toland (1978) and the $2\pi/3$-peaked singularity was proven by Plotnikov (1982).

More recently, numerical and asymptotic results were developed for approximation of nearly-peaked periodic waves and their instabilities.

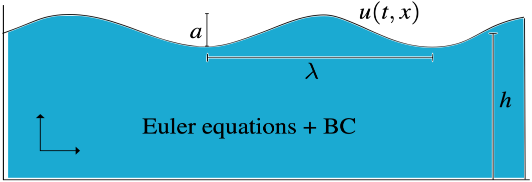
[Dyachenko–Lushnikov–Korotkevich, 2016] [Lushnikov, 2016]

[Dyachenko-Semenova, 23] [Korotkevich-Lushnikov-Semenova-Dyachenko, 23]

Shallow-water models

Shallow water models are derived for long waves of small amplitude

$$a \ll h \ll \lambda$$



Shallow-water models

The Korteweg–de Vries (KdV) equation:

$$u_t + u_x + u_{xxx} + u\, u_x = 0$$

[Boussinesq, 1872] [Korteweg & de Vries, 1895]

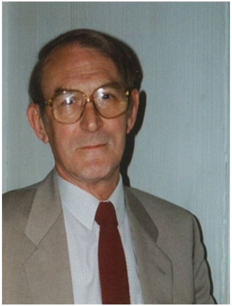
M. Boussinesq.

Shallow-water models

The Benjamin–Bona–Mahony (BBM) equation

$$u_t + u_x - u_{txx} + u\,u_x = 0$$

[Peregrine, 1966]   [Benjamin–Bona–Mahony, 1972]


Shallow-water models

The Camassa–Holm (CH) equation

$$u_t + u_x - u_{txx} + 3\, u\, u_x = 2\, u_x u_{xx} + u\, u_{xxx}$$

[Camassa & Holm, 1993] [Johnson, 2000] [Constantin & Lannes, 2009]

Shallow-water models

The Ostrovsky equation

$$u_t + u_x - u_{txx} + 3\,u\,u_x = \partial_x^{-1} u$$

[Ostrovsky, 1978]

Shallow-water models

Toy model based on holomorphic coordinates

$$2cu_t = (c^2 - 2u)u_x + \partial_x^{-1}\left[u + (u_x)^2\right].$$

[Locke & P, 2024]

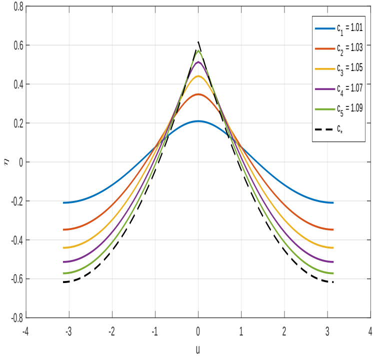
Shallow-water models

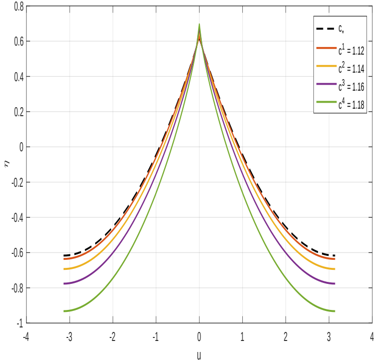
Toy model based on holomorphic coordinates

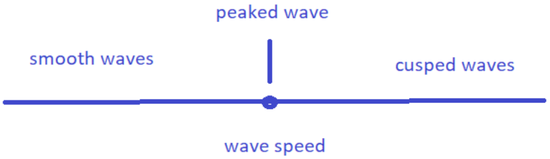
$$2cu_t = (c^2 - 2u)u_x + \partial_x^{-1}\left[u + (u_x)^2\right].$$

[Locke & P, 2024] known as the Hunter–Saxton (HS) equation

Existence and stability of traveling (Stokes) waves







Existence and stability of traveling (Stokes) waves

Standard approach to orbital stability of traveling waves with translation symmetry related to momentum Q and energy H.

- Construct $\Lambda(u) := H(u) + cQ(u)$, such that TW with profile ϕ is a critical point of Λ: $\underbrace{\Lambda'(\phi) = 0}_{\text{TW-eq}}$

- Compute the spectrum of the linearized operator $\mathcal{L} = \Lambda''(\phi)$ and control the negative and zero subspaces of $\mathcal{L}$ in L^2.

- If $\mathcal{L}$ has only one negative simple eigenvalue and a simple zero eigenvalue, then we need to prove that TW is a constrained minimizer of H under fixed Q, i.e. $\mathcal{L}|_{\{Q'(\phi)\}^{\perp}} \geq 0$.

- the orbit of TWs $\{\phi(\cdot + \xi)\}_{\xi \in \mathbb{R}}$ is stable in energy space if local well-posedness has been proven in the energy space.

[A. Geyer & D. P., *Stability of nonlinear waves in Hamiltonian systems*, AMS Monographs, 2025]

Existence and stability of traveling (Stokes) waves

Common features of the KdV and BBM equations:

- Solutions of the initial-value problem exist in Sobolev space $H^1(\mathbb{R})$
- Energy H and momentum Q are defined in $H^1(\mathbb{R})$ and conserved
- Traveling waves $u(x,t) = \phi(x - ct)$ have smooth profiles ϕ in the admissible range of the wave speed c
- TWs are orbitally stable in $H^1(\mathbb{R})$ as constrained minimizers of energy subject to fixed momentum.

Existence and stability of traveling (Stokes) waves

Common features of the CH, Ostrovsky, and HS equations:

- Solutions of the initial-value problem exist in $H^1 \cap W^{1,\infty}$
  [De Lellis–Kappeler-Topalov (2007)] [Linares–Ponce–Sideris (2019)]
- Traveling waves $u(x,t) = \phi(x - ct)$ are smooth only in a subset of parameters and either peaked or cusped outside the subset
  [Lennels (2005)] [Geyer–Martins–Natali–P (2022)]
- Smooth and peaked waves are constrained minimizers of energy
  [Constantin & Strauss, 2000] [Constantin & Molinet, 2001] [Lennels, 2005]
- Waves with smooth profiles are stable in the time evolution
  [Constantin & Strauss, 2002] [Lennels, 2006]
- Waves with peaked profiles are unstable in the time evolution
  [Natali & P., 2020] [Madiyeva & P., 2021] [Lafortune & P., 2022]

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Dynamics of the CH equation

The local differential equation

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx}$$

can be rewritten in the integral form of the perturbed Burgers equation

$$u_t + uu_x + \frac{1}{4}\varphi' * \left(bu^2 + (3-b)u_x^2\right) = 0,$$

where $\varphi := 2(1-\partial_x^2)^{-1}\delta = e^{-|x|}$ is the Green function.

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The time evolution consists of two quadratic parts:

$$\boxed{u_t + uu_x} + \frac{1}{4}\boxed{\varphi' * \left(bu^2 + (3-b)u_x^2\right)} = 0,$$

with Burgers advection $\boxed{u_t + uu_x = 0}$ and convolution smoothing.

Dynamics of the CH equation

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Solutions of the Burgers equation $u_t + uu_x = 0$ with $u(0,x) = f(x)$ admit wave breaking (gradient blowup) for $f \in W^{1,\infty}(\mathbb{R})$:

$$u(t,x) = f(x - tu(t,x)) \quad \Rightarrow \quad u_x = \frac{f'(x-tu)}{1 + tf'(x-tu)}.$$

Dynamics of the CH equation

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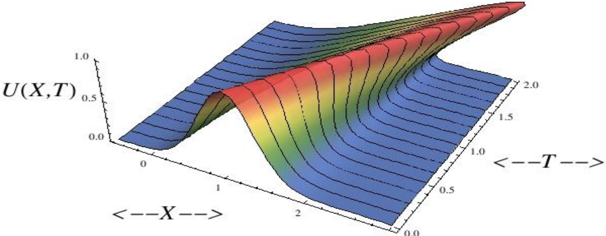

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where $\varphi := 2(1-\partial_x^2)^{-1}\delta = e^{-|x|}$ is the Green function.

We say that the dynamics leads to the wave breaking if

$$\|u(t,\cdot)\|_{L^\infty} < \infty, \quad \|u_x(t,\cdot)\|_{L^\infty} \to \infty \quad \text{as } \; t \to T < \infty$$

Dynamics of the CH equation

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where $\varphi := 2(1-\partial_x^2)^{-1}\delta = e^{-|x|}$ is the Green function.

The initial-value problem is

- locally well-posed in H^s, $s > 3/2$ [Escher & Yin, 2008; Zhou, 2010]
- no continuous dependence in H^s, $s \leq 3/2$ (ill-posed)
  [Himonas, Grayshan, Holliman (2016)] [Guo, Liu, Molinet, Yin (2018)]
- locally well-posed in $H^1 \cap W^{1,\infty}$.
  [De Lellis, Kappeler, Topalov (2007)] [Linares, Ponce, Sideris (2019)]

Existence of traveling waves (peakons)

Smooth traveling waves of the form $u(x,t) = \phi(x - ct)$ satisfy

$$-(c-\phi)(\phi''' - \phi') + b(\phi'' - \phi)\phi' = 0$$

Standard integration gives

$$-(c-\phi)(\phi'' - \phi) + \frac{1}{2}(b-1)((\phi')^2 - \phi^2) = g = \text{const}$$

Alternative integration, after multiplication by $(c-\phi)^{b-1}$, gives

$$-(c-\phi)^b(\phi'' - \phi) = a = \text{const.}$$

Both second-order equations are compatible iff

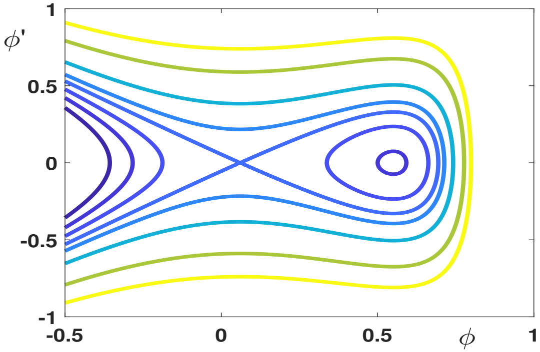
$$\frac{1}{2}(b-1)((\phi')^2 - \phi^2) + \frac{a}{(c-\phi)^{b-1}} = g$$

Existence of traveling waves (peakons)

Analyzing on the phase plane (ϕ, ϕ'),

$$\frac{1}{2}(b-1)((\phi')^2 - \phi^2) + \frac{a}{(c-\phi)^{b-1}} = g$$

e.g., for $b = 3$ and $c = 1$, gives smooth solutions for $a > 0$:

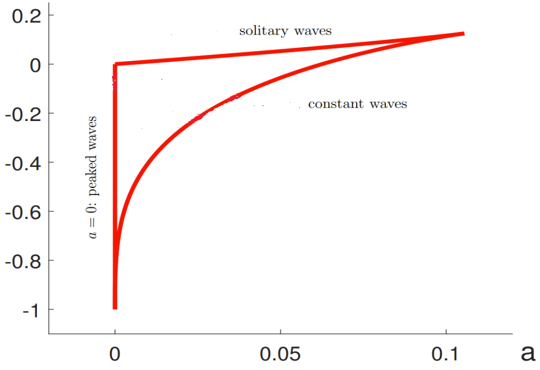


Existence of traveling waves (peakons)

The existence domain of the smooth periodic solutions of

$$\frac{1}{2}(b-1)((\phi')^2 - \phi^2) + \frac{a}{(c-\phi)^{b-1}} = g$$

on the (a, g) plane for fixed $c = 1$:



Existence of traveling waves (peakons)

For peakons, we should use the weak formulation

$$u_t + uu_x + \frac{1}{4}\varphi' * \left(bu^2 + (3-b)u_x^2\right) = 0.$$

After the traveling wave reduction $u(x,t) = \phi(x - ct)$, we obtain the integral equation

$$-c\phi + \frac{1}{2}\phi^2 + \frac{1}{4}\varphi * \left(b\phi^2 + (3-b)(\phi')^2\right) = 0,$$

where $\varphi(x) = e^{-|x|}$.

The peakon $\phi(x) = c\varphi(x)$ is the exact solution of the integral equation. Note that

$$c = \max_{x \in \mathbb{R}} \phi(x).$$

Existence of traveling waves (peakons)

Stumpons were also suggested in the past:

$$u(t,x) = \phi_L(x-ct) = \begin{cases} ce^{-|x-ct|+L}, & |x-ct| > L, \\ c, & |x-ct| \leq L. \end{cases}$$

However, ϕ_L does not satisfy the integral equation for every $L > 0$:

$$-c\phi + \frac{1}{2}\phi^2 + \frac{1}{4}\varphi * \left(b\phi^2 + (3-b)(\phi')^2\right) = 0.$$

[Galtung & Grunert (2022)]

Orbital stability of peakons in $H^1(\mathbb{R})$: $b = 2$

For $b = 2$, the Camassa–Holm equation

$$u_t - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx}$$

has the first three conserved quantities

$$M(u) = \int u dx, \quad E(u) = \frac{1}{2}\int (u^2 + u_x^2) dx, \quad H(u) = \frac{1}{2}\int (u^3 + u u_x^2)\, dx.$$

Orbital stability of peakons in $H^1(\mathbb{R})$: $b = 2$

Theorem (Constantin–Molinet, 2001)

$\varphi = e^{-|x|}$ is a unique (up to translation) minimizer of Hamiltonian $H(u)$ in $H^1(\mathbb{R})$ subject to fixed momentum $E(u)$.

Theorem (Constantin–Strauss, 2000)

For every small $\varepsilon > 0$, if the initial data satisfies

$$\|u_0 - \varphi\|_{H^1} < \left(\frac{\varepsilon}{3}\right)^4,$$

then the solution satisfies

$$\|u(t,\cdot) - \varphi(\cdot - \xi(t))\|_{H^1} < \varepsilon, \quad t \in (0,T),$$

where $\xi(t)$ is a point of maximum for $u(t,\cdot)$.

Yet, we claim instability of peakons in $H^1 \cap W^{1,\infty}$: $b = 2$

Consider solutions of the Cauchy problem:

$$\begin{cases} u_t + uu_x + Q[u] = 0, \\ u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty}, \end{cases} \qquad Q[u] := \frac{1}{4}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right).$$

Theorem (Natali–P., 2020)

For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1 \cap W^{1,\infty}$ satisfying

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta,$$

s.t. the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ with $T > t_0$ satisfies

$$\|u_x(t_0,\cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1,$$

where $\xi(t)$ is a point of peak of $u(t,\cdot)$ for $t \in [0,T)$.

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Consider solutions of the Cauchy problem:

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- If $u \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, then $Q[u]$ is Lipschitz continuous and the method of characteristics can be used to analyze dynamics.
- If there exists a peak at $\xi(t)$ s.t. $u(t,\cdot) \in H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{\xi(t)\})$, then it moves with the local characteristic speed as

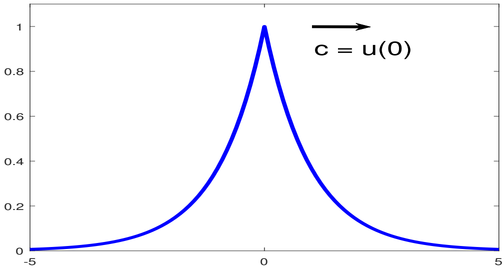
$$\frac{d\xi}{dt} = u(t, \xi(t)), \quad t \in (0, T).$$

Yet, we claim instability of peakons in $H^1 \cap W^{1,\infty}$: $b = 2$

Consider solutions of the Cauchy problem:

$$\begin{cases} u_t + uu_x + Q[u] = 0, \\ u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty}, \end{cases} \qquad Q[u] := \frac{1}{4}\varphi' * \left(u^2 + \frac{1}{2}u_x^2\right).$$

For the peaked traveling wave $u(t,x) = \phi(x - ct)$,
$\xi'(t) = u(t, \xi(t))$ gives $c = \phi(0) := \max_{x \in \mathbb{R}} \phi(x)$.



Evolution of a perturbed peakon

Consider a decomposition near a single peakon:

$$u(t,x) = \varphi(x - t - a(t)) + v(t, x - t - a(t)), \quad t \in [0,T), \quad x \in \mathbb{R},$$

with the peak at $\xi(t) = t + a(t)$ for $v(t,\cdot) \in H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{\xi(t)\})$.

Then, $\xi'(t) = u(t,\xi(t))$ yields $a'(t) = v(t,0)$ and the perturbation $v(t,\cdot)$ satisfies

$$\boxed{v_t = (1-\varphi)v_x + \varphi \int_0^x v(t,y)dy} + \boxed{(v|_{x=0} - v)v_x - Q[v].}$$

Translational invariance is broked at the peak's location.

Nonlinear evolution

For the evolution problem:

$$\begin{cases} v_t = (c-\varphi)v_x + \varphi \int_0^x v(t,y)dy + (v|_{x=0} - v)v_x - Q[v], & t \in (0,T), \\ v|_{t=0} = v_0(x), \end{cases}$$

we can analyze solutions with the method of characteristic curves:

$$x = X(t,s), \qquad v(t, X(t,s)) = V(t,s).$$

Nonlinear evolution

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we can analyze solutions with the method of characteristic curves:

$$x = X(t,s), \qquad v(t, X(t,s)) = V(t,s).$$

The characteristic coordinates $X(t,s)$ satisfies

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - 1 + v(t,X) - v(t,0), \quad t \in (0,T), \\ X|_{t=0} = s. \end{cases}$$

Since φ is Lipschitz, there exists the unique characteristic function $X(t,s)$ for each $s \in \mathbb{R}$ if $v(t,\cdot)$ remains in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$
The peak location $X(t,0) = 0$ is invariant in time.

Nonlinear evolution

For the evolution problem:

$$\begin{cases} v_t = (c-\varphi)v_x + \varphi \int_0^x v(t,y)dy + (v|_{x=0} - v)v_x - Q[v], \quad t \in (0,T), \\ v|_{t=0} = v_0(x), \end{cases}$$

we can analyze solutions with the method of characteristic curves:

$$x = X(t,s), \qquad v(t, X(t,s)) = V(t,s).$$

From the right side of the peak, $V_0(t) = v(t,0)$, $W_0(t) = v_x(t,0^+)$:

$$\frac{dW_0}{dt} = W_0 + V_0 + V_0^2 - \frac{1}{2}W_0^2 - P[v](0), \quad P[v] := \varphi * \left(v^2 + \frac{1}{2}v_x^2\right).$$

We need to show that $W_0(t)$ grows.

Nonlinear instability

From the orbital stability in $H^1(\mathbb{R})$ [A. Constantin, W. Strauss (2000)]
If $\|v_0\|_{H^1} < (\varepsilon/3)^4$, then

$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

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To show instability, we use eq. on the right side of the peak:

$$\frac{dW_0}{dt} = W_0 + V_0 + V_0^2 - \frac{1}{2}W_0^2 - P[v](0)$$

and since $P[v] > 0$, we have

$$\frac{dW_0}{dt} \leq W_0 + C\varepsilon \quad \Rightarrow \quad W_0(t) \leq [W_0(0) + C\varepsilon]\, e^t$$

Nonlinear instability

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$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

If $W_0(0) = -2C\varepsilon$, then

$$W_0(t) \leq -C\varepsilon e^t,$$

hence $|W_0(t_0)| \geq 1$ for $t_0 := -\log(C\varepsilon)$.

Nonlinear instability

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The initial constraint $\|v_0\|_{L^\infty} + \|v_0'\|_{L^\infty} < \delta$, is satisfied
if $\forall\delta > 0$, $\exists\varepsilon > 0$ such that

$$\left(\frac{\varepsilon}{3}\right)^4 + 2C\varepsilon < \delta.$$

Linear instability

For the linearized equation $\boxed{v_t = (1-\varphi)v_x + \varphi \int_0^x v(t,y)dy}$, we can obtain exact unstable solutions (Madiyeva & P, 2021) satisfying

$$C_- e^t \leq \|v_x(t,\cdot)\|_{L^\infty(0,\infty)} \leq C_+ e^t$$

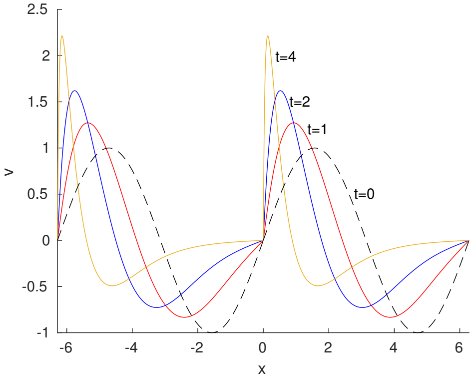

Figure 1: Perturbation $v(t,x)$ versus x for $t = 0, 1, 2, 4$ with $v(0,x) = \sin(x)$.

Spectral instability of peakons

For the b-CH equation, the linearized equation is well-posed in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$:

$$\boxed{v_t = (1-\varphi)v_x + (b-2)(v|_{x=0} - v)\varphi'} + \frac{1}{2}\boxed{(b-3)\varphi * (\varphi' v) - (2b-3)\varphi' * (\varphi v)},$$

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The linearized operator is

$$L = (1-\varphi)\partial_x - (b-2)\varphi' + K,$$

where $K : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a compact (Hilbert–Schmidt) operator. Since $\varphi \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, the natural domain of L in $L^2(\mathbb{R})$ is

$$\text{Dom}(L) = \left\{ v \in L^2(\mathbb{R}) : \quad (1-\varphi)v' \in L^2(\mathbb{R}) \right\}.$$

Spectrum of a linear operator

Theorem (S. Lafortune–D. P, 2022)

The spectrum of L *with* $\mathrm{Dom}(L) \subset L^2(\mathbb{R})$

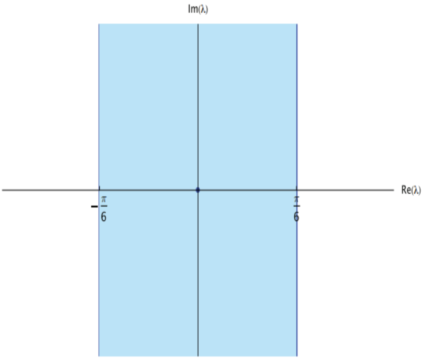
$$\sigma(L) = \left\{ \lambda \in \mathbb{C} : \quad |\mathrm{Re}(\lambda)| \leq \left| \frac{5}{2} - b \right| \right\}.$$

Moreover,

- $\sigma_p(L)$ *is located for* $0 < |\mathrm{Re}(\lambda)| < \frac{5}{2} - b$ *if* $b < \frac{5}{2}$
- $\sigma_r(L)$ *is located for* $0 < |\mathrm{Re}(\lambda)| < b - \frac{5}{2}$ *if* $b > \frac{5}{2}$
- $\sigma_c(L)$ *is located for* $\mathrm{Re}(\lambda) = 0$ *and* $\mathrm{Re}(\lambda) = \pm \left| \frac{5}{2} - b \right|$.

$\Rightarrow$ the peakon is linearly unstable in $\mathrm{Dom}(L)$ for every $b \neq \frac{5}{2}$.

Spectrum of a linear operator



The width of the strip is $|b - \frac{5}{2}|$ in $L^2(\mathbb{R})$. If the operator L is defined in $H^s(\mathbb{R})$, the width is decreasing for higher $s \geq 0$.

[S. Charalampidis, R. Parker, P. Kevrekidis, S. Lafortune, (2023)]

First results with instability in the vertical strip were derived for Euler flows [R. Shvidkoy, Yu. Latushkin (2003)]

The main tool for the spectral instability

Recall that $L = L_0 + K$, where $L_0 := (1-\varphi)\partial_x - (b-2)\varphi'$ with

$$\mathsf{Dom}(L) = \mathsf{Dom}(L_0) = \left\{ v \in L^2(\mathbb{R}) : \quad (1-\varphi)v' \in L^2(\mathbb{R}) \right\}$$

and $K : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a compact (Hilbert–Schmidt) operator.

The truncated spectral problem $L_0 v = \lambda v$ is the first-order equation

$$(1-\varphi)\frac{dv}{dx} + (2-b)\varphi' v = \lambda v$$

with the exact solution

$$v(x) = \begin{cases} v_+ e^{\lambda x}(1-e^{-x})^{2+\lambda-b}, & x > 0, \\ v_- e^{\lambda x}(1-e^{x})^{2-\lambda-b}, & x < 0, \end{cases}$$

If $\mathrm{Re}(\lambda) > 0$, then $v_+ = 0$ and $\mathrm{Re}(\lambda) < \frac{5}{2} - b$.

Similar solutions can be found for $L_0^* v = \lambda v$.

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Euler equations in physical coordinates

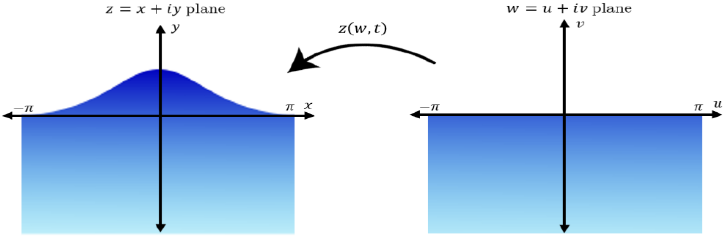
- $\eta(x,t)$ - the free surface profile.
- $\phi(x,y,t)$ - velocity potential satisfying the Laplace equation in

$$D_\eta(t) := \{(x,y): \quad -\pi \le x \le \pi, \quad -h_0 \le y \le \eta(x,t)\}$$

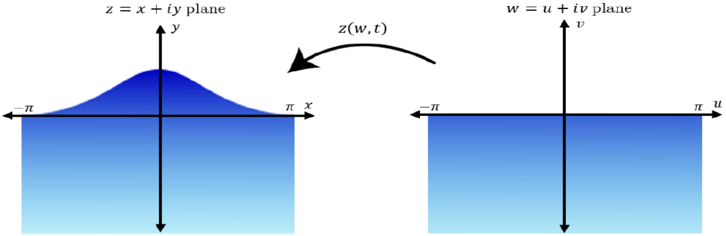
- Periodic boundary conditions at $x = \pm\pi$.
- Neumann boundary condition $\varphi_y|_{y=-h_0} = 0$.
- Nonlinear evolution equatons at the free surface:

$$\left.\begin{aligned} \eta_t + \varphi_x \eta_x - \varphi_y = 0, \\ \varphi_t + \frac{1}{2}(\varphi_x)^2 + \frac{1}{2}(\varphi_y)^2 + \eta = 0, \end{aligned}\right\} \qquad \text{at } \ y = \eta(x,t),$$

Conformal transformation



Conformal transformation



The velocity potential is uniquely represented by

$$\varphi(u,v,t) = \sum_{n\in\mathbb{Z}} \hat{\xi}_n(t) e^{inu} \frac{\cosh(n(v+h))}{\cosh(nh)},$$

where $\hat{\xi}_n(t)$ is the Fourier coefficient for $\xi(u,t) = \varphi(u, v=0, t)$.
The other canonical variable is $\eta(u,t) = y(u, v=0, t)$.

Evolution equations for $\xi(u,t)$ and $\eta(u,t)$

The closed system of two evolution equations in holomorphic variables is

$$\left\{ \begin{array}{l} (1+K_h\eta)\eta_t - \eta_u T_h^{-1}\eta_t + T_h\xi_u = 0, \\ \xi_t\eta_u - \xi_u\eta_t + \eta\eta_u + T_h\left[(1+K_h\eta)\xi_t - \xi_u T_h^{-1}\eta_t + (1+K_h\eta)\eta\right] = 0, \end{array} \right.$$

where skew-adjoint operators T_h and T_h^{-1} are defined by

$$\widehat{(T_h)}_n = i\tanh(hn), \quad n\in\mathbb{Z}, \qquad \widehat{(T_h^{-1})}_n = \left\{ \begin{array}{ll} -i\coth(hn), & n\in\mathbb{Z}\backslash\{0\}, \\ 0, & n=0, \end{array} \right.$$

whereas the self-adjoint operator $K_h = T_h^{-1}\partial_u$ is defined by

$$\widehat{(K_h)}_n = \left\{ \begin{array}{ll} n\coth(hn), & n\in\mathbb{Z}\backslash\{0\}, \\ 0, & n=0. \end{array} \right.$$

[Dyachenko-elder–Kuznetsov–Spector–Zakharov, 1996]

[Dyachenko-junior–Lushnikov–Korotkevich, 2016]

[Hunter–Ifrim–Tataru, 2016]

Evolution equations for $\xi(u,t)$ and $\eta(u,t)$

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Traveling waves $\eta(u,t) = \eta(u-ct)$ satisfy $\xi = cT_h^{-1}\eta$, where the profile η is a solution of Babenko's equation:

$$\boxed{(c^2K_h - 1)\eta = \frac{1}{2}K_h\eta^2 + \eta K_h\eta.}$$

Both smooth and peaked profiles for 2π-periodic traveling waves are solutions of this scalar equation.

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The closed system of two evolution equations in holomorphic variables is

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If $\eta(u,t) = \eta(u-ct,t)$ and $\xi = cT_h^{-1}\eta + \zeta$, the system can be simplified into the form:

$$(1+K_h\eta)\eta_t - \eta_u T_h^{-1}\eta_t + T_h\zeta_u = 0$$

and

$$(1+K_h\eta)\zeta_t - \zeta_u T_h^{-1}\eta_t + T_h^{-1}(\zeta_t\eta_u - \zeta_u\eta_t)$$

$$\boxed{+2cT_h^{-1}\eta_t - c^2K_h\eta + (1+K_h\eta)\eta + \frac{1}{2}K_h\eta^2 = 0.}$$

Evolution equations for $\xi(u,t)$ and $\eta(u,t)$

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$$\left\{\begin{array}{l} (1+K_h\eta)\eta_t - \eta_u T_h^{-1}\eta_t + T_h\xi_u = 0, \\ \xi_t\eta_u - \xi_u\eta_t + \eta\eta_u + T_h\left[(1+K_h\eta)\xi_t - \xi_u T_h^{-1}\eta_t + (1+K_h\eta)\eta\right] = 0, \end{array}\right.$$

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$$\boxed{+2cT_h^{-1}\eta_t - c^2K_h\eta + (1+K_h\eta)\eta + \frac{1}{2}K_h\eta^2 = 0.}$$

The local model arises when we take $K_h = -\partial_u^2$ and $T_h^{-1} = -\partial_u$:

$$\boxed{2c\partial_u\partial_t\eta = (c^2 - 2\eta)\partial_u^2\eta - (\partial_u\eta)^2 + \eta.}$$

Conserved quantities

Thus, we can consider the toy model (the Hunter–Saxton equation):

$$\boxed{2c\partial_u\partial_t\eta = (c^2 - 2\eta)\partial_u^2\eta - (\partial_u\eta)^2 + \eta}$$

in the 2π-periodic domain $\mathbb{T}$.

The toy model has the first three conserved quantities

$$\oint \eta du, \quad \oint (\partial_u\eta)^2 du, \quad \oint \left[\eta^2 + 2\eta(\partial_u\eta)^2\right] du$$

and the constraint

$$\oint \left[\eta + (\partial_u\eta)^2\right] du = 0,$$

and which is equivalent to the normalization $\oint \eta dx = 0$ in x-variable.

Local well-posedness of the initial-value problem

The toy model

$$\boxed{2c\partial_u\partial_t\eta = (c^2 - 2\eta)\partial_u^2\eta - (\partial_u\eta)^2 + \eta}$$

can be rewritten in the week form:

$$2c\partial_t\eta = (c^2 - 2\eta)\partial_u\eta + \Pi_0\partial_u^{-1}\Pi_0\left[(\partial_u\eta)^2 + \eta\right]$$

subject to the constraint $\oint\left[\eta + (\partial_u\eta)^2\right]du = 0$. The inviscid Burgers equation

$$2c\partial_t\eta = (c^2 - 2\eta)\partial_u\eta$$

is locally well-posed in $H^1_{\rm per} \cap W^{1,\infty}$ and the mapping

$$\Pi_0\partial_u^{-1}\Pi_0\left[(\partial_u\eta)^2 + \eta\right] : H^1_{\rm per} \cap W^{1,\infty} \to H^1_{\rm per} \cap W^{1,\infty}$$

is bounded on every bounded subset.

Local well-posedness of the initial-value problem

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subject to the constraint $\oint \left[\eta + (\partial_u\eta)^2\right] du = 0$.

Hence we get by standard technique (e.g. via characteristics)

Theorem (S. Locke–D.P., 2024)

The initial-value problem is locally well-posed in $H^1_{\mathrm{per}} \cap W^{1,\infty}$.

Existence of the periodic wave solutions

If $\eta(u,t) = \eta(u)$ in the traveling wave frame, then η is a solution of the differential equation

$$\boxed{(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0, \qquad u \in \mathbb{T}.}$$

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Theorem (S. Locke–D. P., 2024)

There exist $c_* := \frac{\pi}{2\sqrt{2}}$ *and* $c_\infty \in (c_*, \infty)$ *such that the ODE admits a unique solution with the profile* $\eta \in C^\infty_{\rm per}(\mathbb{T})$ *for every* $c \in (1, c_*)$ *s.t.*

$$\|\eta\|_{L^\infty} \to 0 \quad \text{as} \quad c \to 1$$

and a solution with the profile $\eta \in C^0_{\rm per}(\mathbb{T})$ *for every* $c \in (c_*, c_\infty)$ *satisfying for some* $A(c) > 0$*,*

$$\eta(u) = \frac{c^2}{2} - A(c)|u|^{2/3} + \mathcal{O}(|u|^{4/3}) \quad \text{as} \quad u \to 0.$$

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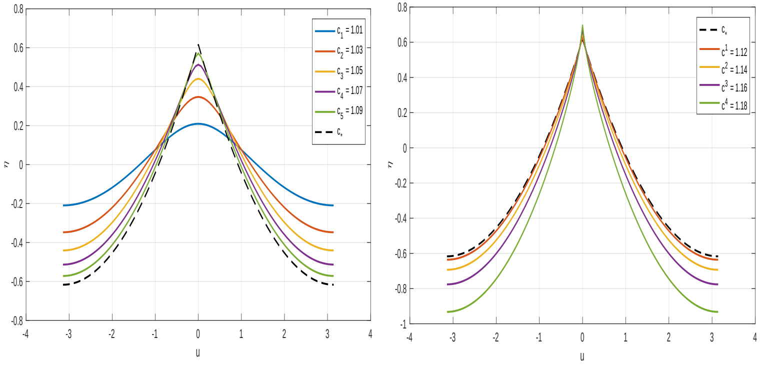

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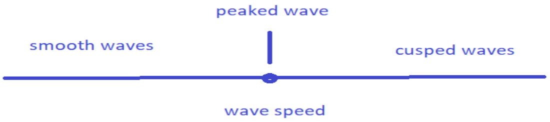
$$\boxed{(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0, \qquad u \in \mathbb{T}.}$$

The two continuous families meet at $c = c_*$, where the peaked

profile is explicit:

$$\eta(u) = \frac{1}{16}(\pi^2 - 4\pi|u| + 2u^2), \qquad u \in \mathbb{T}.$$



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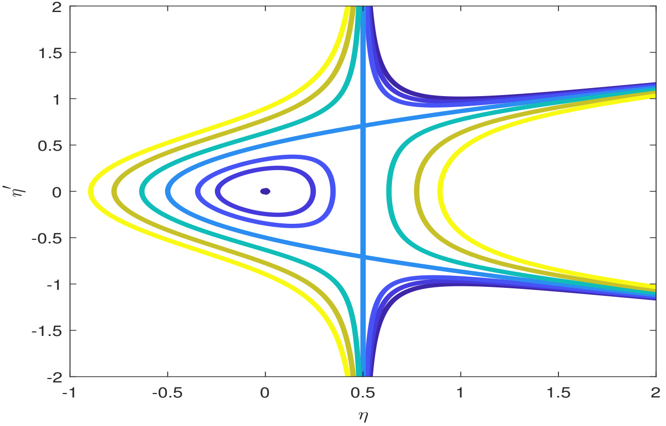
Note that the highest amplitude

$$\max_{u \in \mathbb{T}} \eta(u) = \eta(0) = \frac{c^2}{2}$$

follows from Bernoulli's principle of hydrodynamics and that the $|u|^{2/3}$ singularity corresponds after the conformal transformation to Stokes' law of the 120^0 angle in the physical coordinate.

Existence of the periodic wave solutions



Linear stability of periodic waves with smooth profile

By substituting $\eta(u) + \hat{\eta}(u,t)$ into

$$2c\partial_t\eta = (c^2 - 2\eta)\partial_u\eta + \partial_u^{-1}\left[(\partial_u\eta)^2 + \eta\right]$$

we obtain the linearized equation with $\hat{\eta}$:

$$2c\partial_t\hat{\eta} = -\partial_u^{-1}\mathcal{L}\hat{\eta}, \qquad \mathcal{L} = -\partial_u(c^2 - 2\eta)\partial_u - 1 + 2\eta''.$$

Linear stability of periodic waves with smooth profile

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TW with the smooth profile η is a constrained minimizer of

$$\oint \left[\eta^2 + 2\eta(\partial_u \eta)^2\right] du \ \text{ for fixed } \oint \eta du \quad \text{and} \quad \oint (\partial_u \eta)^2 du$$

so that it is linearly stable.

[Locke,P, 2024] [Stanislovova–Stefanov, 2016]

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$$2c\partial_t \hat{\eta} = -\partial_u^{-1} \mathcal{L} \hat{\eta}, \qquad \mathcal{L} = -\partial_u (c^2 - 2\eta)\partial_u - 1 + 2\eta''.$$

The peaked wave for $c = c_*$ is linearly unstable.

$$\eta(u) = \frac{1}{16}(\pi^2 - 4\pi|u| + 2u^2), \qquad u \in \mathbb{T}.$$

[P., Wang, in progress]

Table of contents

Existence of traveling waves

For a similar model of the reduced Ostrovsky equation

$$u_t + uu_x = \partial_x^{-1} u,$$

smooth traveling wave solutions in the form $u(x,t) = \phi(x - ct)$ satisfy

$$\frac{d}{dx}\left((c-\phi)\frac{d\phi}{dx}\right) + \phi(x) = 0, \quad x \in \mathbb{T},$$

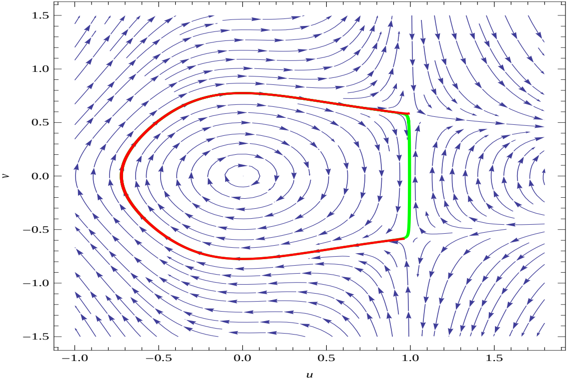
under the zero-mean constraint $\oint \phi(x)dx = 0$.

Existence of traveling waves

For a similar model of the reduced Ostrovsky equation

$$u_t + uu_x = \partial_x^{-1} u,$$

The first integral is $E(\phi, \phi') = \frac{1}{2}(c-\phi)^2(\phi')^2 + \frac{c}{2}\phi^2 - \frac{1}{3}\phi^3.$



Existence of traveling waves

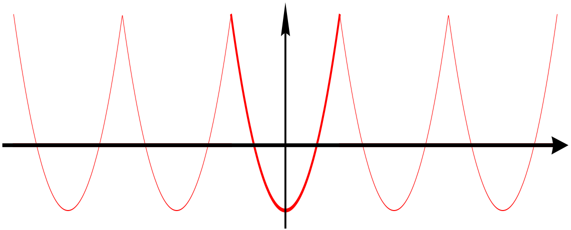
For a similar model of the reduced Ostrovsky equation

$$u_t + uu_x = \partial_x^{-1} u,$$

For $c = c_* := \frac{\pi^2}{9}$ the peaked wave has the parabolic profile

$$\phi(x) = \frac{3x^2 - \pi^2}{18}, \quad x \in \mathbb{T},$$

which can be periodically continued as the peaked periodic wave.



Existence of traveling waves

For a similar model of the reduced Ostrovsky equation

$$u_t + uu_x = \partial_x^{-1} u,$$

Uniqueness of the peaked periodic wave for $c = c_*$ was proven in
[A. Geyer & D.P, 2019] [G. Bruell & Dhara, 2019]

Interesting that cusped profiles do not exist in the weak formulation for $c > c_*$ [A. Geyer & D.P, 2019].

Stability of smooth traveling waves

Using $u(x,t) = \phi(x-ct) + v(x-ct)e^{\lambda t}$, one can obtain the spectral stability problem in the form

$$\boxed{\lambda v = \partial_x \mathcal{L} v}$$

with the self-adjoint linear operator

$$\mathcal{L} = \Pi_0 \left(\partial_x^{-2} + c - \phi \right) \Pi_0 : \; \dot{L}^2_{\rm per} \to \dot{L}^2_{\rm per},$$

where $\dot{L}^2_{\rm per}$ is the L^2 space of periodic function with zero mean.

Spectral stability of smooth periodic waves was proven in
[Hakkaev & Stanislavova & Stefanov, 2017] [Johnson & P., 2016] [A. Geyer & P., 2017]

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where $\dot{L}^2_{\rm per}$ is the L^2 space of periodic function with zero mean.

The smooth periodic wave with the profile ϕ is a local constrained minimizer of the energy $H(u)$ subject to the fixed momentum $Q(u)$:

$$H(u) = -\frac{1}{2} \oint (\partial_x^{-1} u)^2 dx - \frac{1}{6} \oint u^3 dx, \quad Q(u) = \frac{1}{2} \oint u^2 dx$$

with $\mathcal{L}$ being the Hessian of $H(u) + cQ(u)$.

Stability of smooth traveling waves

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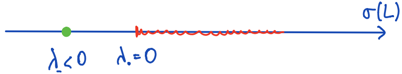
where $\dot{L}^2_{\rm per}$ is the L^2 space of periodic function with zero mean.

The stability argument breaks in the limit $c \to c_*$, where the smooth profile becomes peaked.



[A. Geyer & D.P, 2019].

Linear instability of the peaked periodic wave

Linearized evolution the perturbation v to the peaked profile ϕ_*:

$$\begin{cases} v_t + \partial_x \left[(\phi_*(x) - c_*) v \right] = \partial_x^{-1} v, & t > 0, \\ v|_{t=0} = v_0. \end{cases}$$

Theorem (A. Geyer, D.P., 2019)

For every $v_0 \in \mathrm{Dom}(\partial_x \mathcal{L})$ $\exists!$ global solution $v \in C^0(\mathbb{R}, \mathrm{Dom}(\partial_x \mathcal{L}))$. If v_0 is odd, then the solution satisfies

$$C \|v_0\|_{L^2} e^{\pi t/6} \leq \|v(t,\cdot)\|_{L^2} \leq \|v_0\|_{L^2} e^{\pi t/6}, \quad t > 0,$$

which implies the linear instability of the profile ϕ_*.

Linear instability of the peaked periodic wave

For the spectral problem

$$\lambda v = Av := \partial_x \left[(c_* - \phi_*(x))v\right] + \partial_x^{-1} v,$$

with

$$\text{Dom}(A) = \left\{ v \in \dot{L}^2_{\text{per}} : \quad \partial_x \left[(c_* - \phi_*)v\right] \in \dot{L}^2_{\text{per}} \right\}.$$

Theorem (A. Geyer & D. P., 2020)

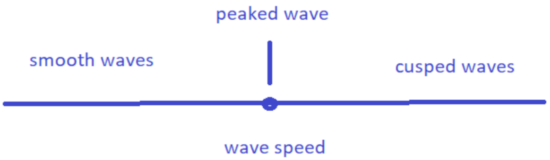
$$\sigma(A) = \left\{ \lambda \in \mathbb{C} : \quad -\frac{\pi}{6} \leq \text{Re}(\lambda) \leq \frac{\pi}{6} \right\}.$$

The width of the instability band corresponds to the bound:

$$\frac{1}{2}\|v_0\|_{L^2} e^{\pi t/6} \leq \|v(t,\cdot)\|_{L^2} \leq \|v_0\|_{L^2} e^{\pi t/6}, \quad t > 0.$$

Summary

Three different models with peaked waves admit the same pattern:



- The smooth waves are linearly stable in the time evolution.
- The peaked wave is linearly unstable in the time evolution.
- The initial-value problem is locally well-posed in $H^1 \cap W^{1,\infty}$, which excludes the family of cusped waves.