Dispersive hydrodynamics in the modified KdV equation

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Emergent phenomena in nonlinear dispersive waves

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Section 1

Motivations and the state-of-art for the KdV equation:

- stability of the traveling wave background
- breathers on traveling wave background

Motivations

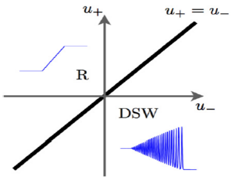
Dispersive hydrodynamics for the canonical model of the Korteweg–de Vries (KdV) equation has been well studied.

$$u_t + 6uu_x + u_{xxx} = 0,$$

with the step-like data

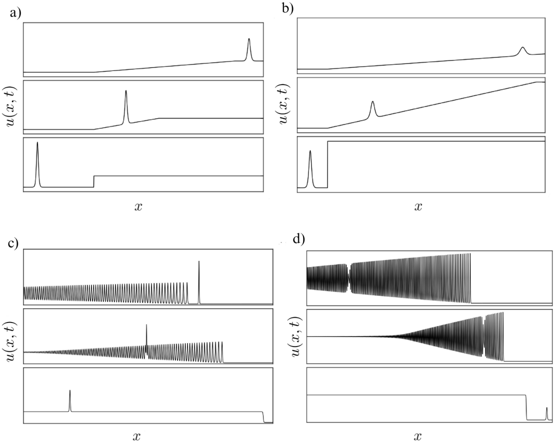
$$\lim_{x \to -\infty} u(t,x) = u_-, \qquad \lim_{x \to +\infty} u(t,x) = u_+.$$

The step-like initial data results in the appearance of a rarefaction wave (RW) if $u_+ > u_-$ and a dispersive shock wave (DSW) if $u_+ < u_-$.



Motivations

Solitons interactions with RWs and DSWs:



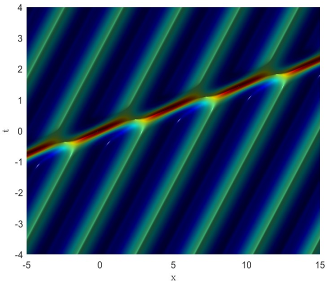
Soliton-RW:
a) Tunneling.
b) Trapping.

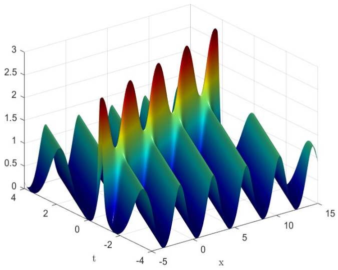
Soliton-DSW:
a) Tunneling.
b) Trapping.

M. J. Ablowitz, J. T. Cole, G. A. El, M. A. Hoefer, X. Luo, Stud. Appl. Math. (2023)

Motivations

Bright breathers on the modulationally stable TW of the KdV equation $u_t + 6uu_x + u_{xxx} = 0$





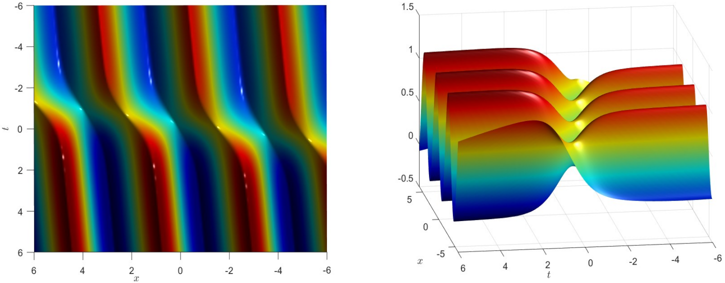
Move faster that the periodic wave and induce the phase shift.

M. Hoefer, A. Mucalica, D.P., JPA (2023)

Motivations

Dark breathers on the modulationally stable TW of the KdV equation

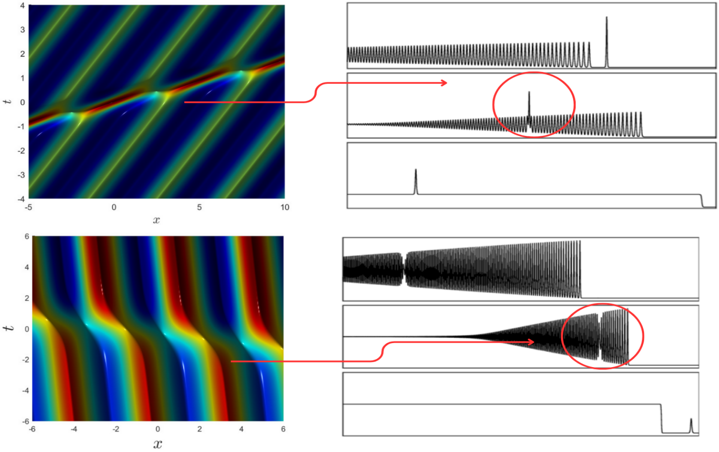
$$u_t + 6uu_x + u_{xxx} = 0.$$



Move slower that the periodic wave and induce the phase shift.

M. Hoefer, A. Mucalica, D.P , JPA (2023)

Motivations



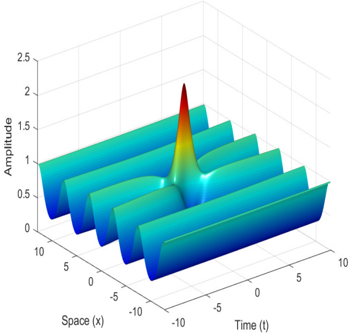
Further applications:

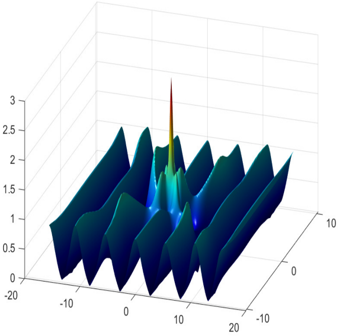
Y. Mao, S. Chandramouli, W. Xu, M. Hoefer, Phys. Rev. Lett. (2023)

Motivations

Rogue waves on the modulationally unstable elliptic background for another canonical model of the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} + |\psi|^2\psi = 0.$$

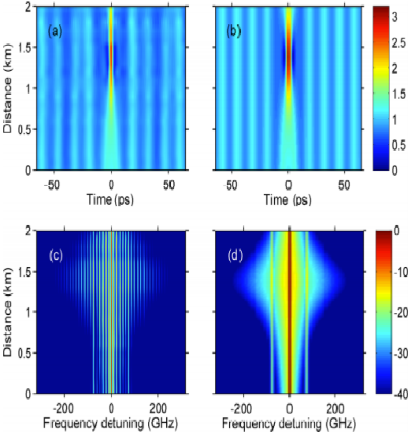


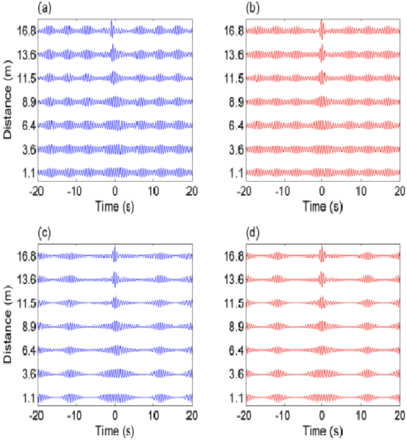


J. Chen, D. P., Proceedings A (2018), J. Chen, D. P., R. White, Physica D (2020)

Motivations

These rogue waves are observed in optics and hydrodynamics:





G. Xu, A. Chabchoub, D.P., B. Kibler, Physical Review Research (2020)

State-of-art for construction of breathers

Traveling waves $u(x,t) = \phi(x - ct)$ of the KdV equation satisfy

$$u_t + 6uu_x + u_{xxx} = 0 \quad \Rightarrow \quad \phi''' + 6\phi\phi' - c\phi' = 0.$$

After integration(s), it yields

$$\phi'' + 3\phi^2 - c\phi = b \quad \Rightarrow \quad (\phi')^2 + 2\phi^3 - c\phi^2 = 2b\phi + d$$

with three parameters (b, c, d). Due to scaling transformation

$$u(x,t) \mapsto \alpha u(\alpha x, \alpha^3 t)$$

and Galilean transformation

$$u(x,t) \mapsto \beta + u(x - 6\beta t, t),$$

only one parameter is independent.

State-of-art for construction of breathers

The normalized form of TW solutions is

$$\phi(x) = 2k^2 \mathrm{cn}^2(x;k), \quad b_0 = 4k^2(1-k^2), \quad c_0 = 4(2k^2-1), \quad d_0 = 0.$$

A general TW solution is obtained by the scaling and Galilean transformations.

sn, cn, and dn are real-valued Jacobi elliptic functions on $\mathbb{R}$ with

$$\mathrm{sn}^2(x;k) + \mathrm{cn}^2(x;k) = 1, \quad \mathrm{dn}^2(x;k) + k^2\mathrm{sn}^2(x;k) = 1,$$

with elliptic modulus $k \in (0,1)$ between $k \to 0$ (trigonometric functions) and $k \to 1$ (hyperbolic functions).

Elliptic functions sn^2, cn^2, and dn^2 are double-periodic in $\mathbb{C}$ with periods $2K(k)$ and $2iK'(k)$, where $K'(k) = K(\sqrt{1-k^2})$.

State-of-art for construction of breathers

Breathers of the KdV equation have been studied before:

E. Kuznetsov, A. Mikhailov, JETP **40** (1974) 855
F. Gesztesy, R. Svirsky, Memoirs AMS **118** (1995) 1–88
X.R. Hu, S.Y. Lou, Y. Chen, Phys. Rev. E **85** (2012) 056607
A. Nakayashiki, Lett. Math. Phys. **111** (2021) 85

with some recent additions:

M. Hoefer, A. Mucalica and D.E. Pelinovsky, KdV breathers on cnoidal wave background, J. Physics A: Mathem. Theor. 56 (2023) 185701

] M. Bertola, R. Jenkins, A. Tovbis, Partial degeneration of finite gap solutions to the KdV equation: soliton gas and scattering on elliptic background, Nonlinearity 36 (2023) 3622?3660

State-of-art for construction of breathers

Our method of construction is the Darboux–Backlund transformation

$$\hat{u} := u + 2\frac{\partial^2}{\partial x^2}\log(v_0),$$

where $u(x,t) = \phi(x - ct)$ is the TW solution of the KdV equation and $v_0(x,t) = v(x - ct)e^{\omega t}$ is a solution of the Lamé equation

$$v''(x) + 2k^2\mathrm{cn}^2(x;k)v(x) + \lambda v(x) = 0$$

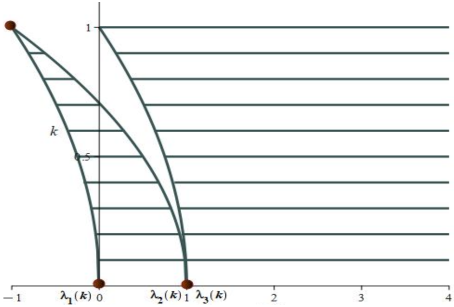
with some uniquely determined $\omega = \omega(\lambda)$ from the linear equation

$$\omega v(x) - cv'(x) = -3\phi'(x)v(x) - 6\phi v'(x) - 4v'''(x).$$

The linear equations represent the Lax pair

$$\mathcal{L}(u)v = \lambda v, \quad \frac{\partial v}{\partial t} = \mathcal{M}(u)v.$$

State-of-art for construction of breathers



Bright breathers correspond to λ in semi-infinite gap.

Dark breathers correspond to λ in the finite gap.

Construction of breathers

The Lamé equation for a given $\lambda \in \mathbb{R}$

$$v''(x) + 2k^2 \text{cn}^2(x;k) v(x) + \lambda v(x) = 0$$

is solved with the explicit functions

$$v_\pm(x) = \frac{H(x \pm \alpha)}{\Theta(x)} e^{\mp x Z(\alpha)}, \quad \lambda = 1 - 2k^2 + k^2 \text{cn}(\alpha;k)$$

where $\alpha \in \mathbb{C}$ is a new parameter and Jacobi' theta functions are

$$H(x) = \theta_1(\frac{\pi x}{2K}) = 2 \sum_{n=1}^{\infty} (-1)^{n-1} q^{(n-1/2)^2} \sin(2n-1)(\frac{\pi x}{2K}),$$

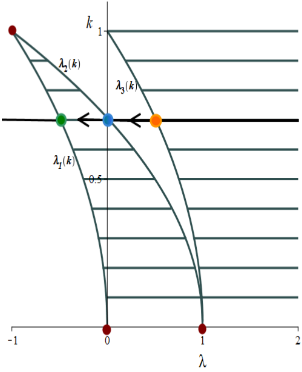
$$\Theta(x) = \theta_4(\frac{\pi x}{2K}) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2n)(\frac{\pi x}{2K}),$$

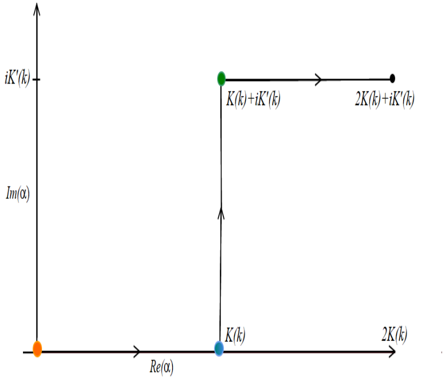
such that $H(0) = 0$ and $\Theta(x) > 0$ for all $x \in \mathbb{R}$.

Construction of breathers

The path in α-complex plane from λ-real line:

$$\lambda = 1 - 2k^2 + k^2 \mathrm{cn}(\alpha; k)$$





Construction of breathers

The time evolution of the eigenfunctions follows from the separation of variables in the Lax system:

$$v_{\pm}(t,x) = \frac{H(x - ct \pm \alpha)}{\Theta(x - ct)} e^{\mp(x-ct)Z(\alpha) \mp t\omega(\alpha)},$$

where $\omega(\alpha)$ is found at $x = 0$:

$$\omega(\alpha) = 4(\lambda + k^2 - 1)\left[\frac{\Theta'(\alpha)}{\Theta(\alpha)} - \frac{H'(\alpha)}{H(\alpha)}\right].$$

Construction of breathers

Darboux transformation for λ in the semi-infinite gap is applied with

$$v_0(x,t) = c_+ v_+(x,t) + c_- v_-(x,t)$$

where $v_\pm(x,t) > 0$ and $c_\pm > 0$. We can use that

$$k^2 \mathrm{cn}^2(x,k) = k^2 - 1 + \frac{E(k)}{K(k)} + \partial_x^2 \log \Theta(x)$$

and obtain the new solution

$$\hat{u} = u + 2\frac{\partial^2}{\partial x^2}\log(v_0) = 2\left[k^2 - 1 + \frac{E(k)}{K(k)}\right] + 2\partial_x^2 \log \tau,$$

$$\tau = \Theta(x - c_0 t + \alpha_b) e^{\kappa_b (x - c_b t + x_0)} + \Theta(x - c_0 t - \alpha_b) e^{-\kappa_b (x - c_b t + x_0)}$$

with uniquely defined parameters $c_b > c_0$, $\kappa_b > 0$, and $\alpha_b \in [0, K]$. This yields the bright breather.

Construction of breathers

Darboux transformation for λ in the finite gap is applied with

$$v_0(x,t) = c_+ v_+(x,t) + c_- v_-(x,t)$$

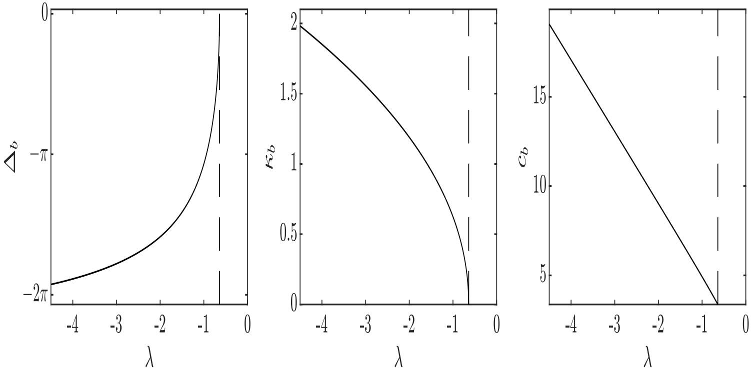
but $v_\pm(x,t)$ are sign-indefinite. However, translation of the new solution $\hat{u} = \hat{u}(x + iK', t)$ yields a bounded solution

$$\hat{u} = u + 2\frac{\partial^2}{\partial x^2}\log(v_0) = 2\left[k^2 - 1 + \frac{E(k)}{K(k)}\right] + 2\partial_x^2 \log\tau,$$

$$\tau = \Theta(x - c_0 t + \alpha_d)e^{-\kappa_d(x - c_d t + x_0)} + \Theta(x - c_0 t - \alpha_d)e^{\kappa_d(x - c_d t + x_0)}$$

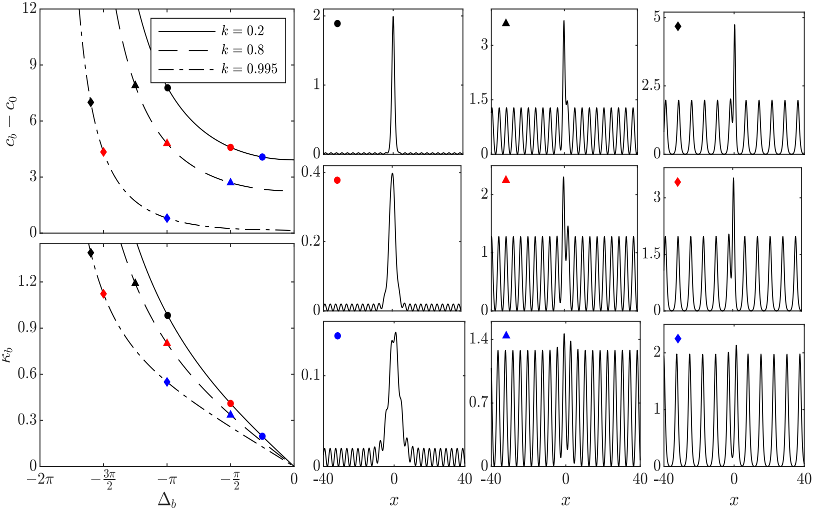
with uniquely defined parameters $c_d < c_0$, $\kappa_d > 0$, and $\alpha_d \in [0, K]$. This yields the dark breather.

Bright breathers

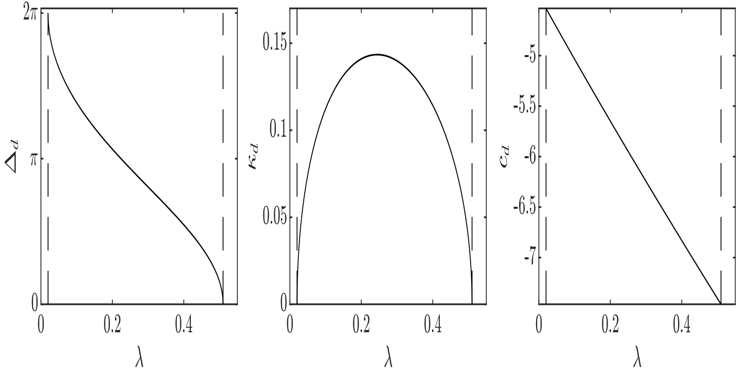


Here $\Delta_b = 2\pi\alpha_b/K(k)$ is normalized phase shift.
We can prove $\Delta_b'(\lambda) > 0$, $\kappa_b'(\lambda) < 0$, and $c_b > c_0$.

Bright breathers

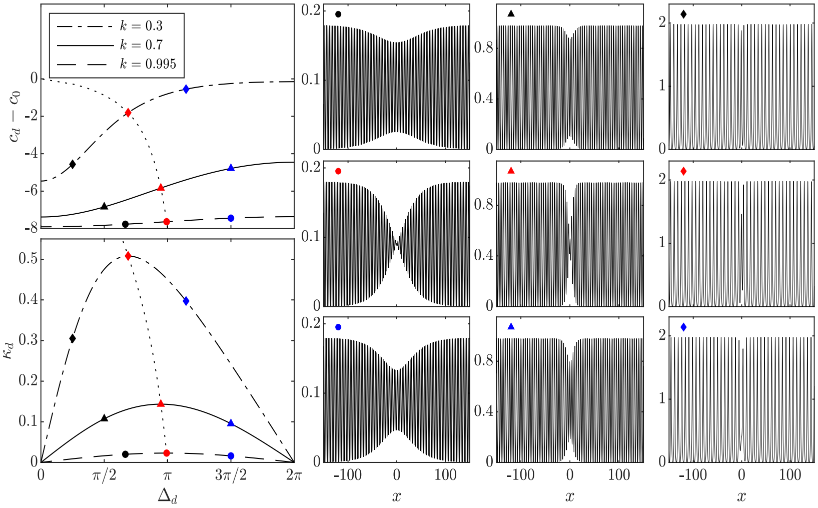


Dark breathers



Here $\Delta_d = 2\pi\alpha_d/K(k)$ is normalized phase shift.
We can prove $\Delta_d'(\lambda) < 0$, $\max \kappa_d(\lambda)$, and $c_d < c_0$.

Dark breathers



Breathers in the Benjamin–Ono equation

The Benjamin–Ono (BO) equation is very similar to the KdV equation as it is derived for stable fluids:

$$u_t + 2uu_x + H(u_{xx}) = 0, \quad H(f) := \frac{1}{\pi}\text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)dy}{y-x}.$$

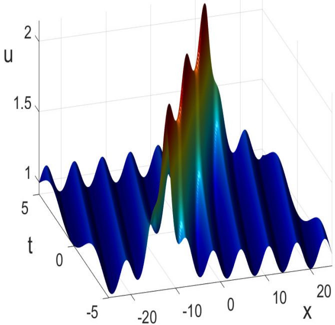
The traveling periodic waves are expressed in elementary functions:

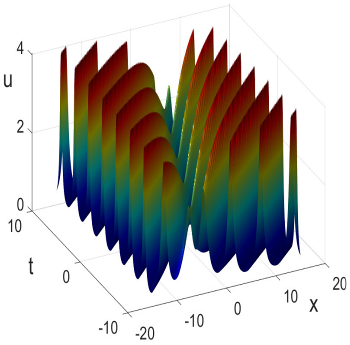
$$u(x,t) = \frac{k \sinh \phi}{\cos(k\xi) + \cosh \phi}, \quad \xi = x - ct - \xi_0, \quad c = k \coth \phi.$$

The Lax spectrum is $[\lambda_0, \lambda_0 + k] \cup [0, \infty)$ with $\lambda_0 := -\frac{c+k}{2}$.

Dobrokhotov & Krichever (1991); Gérard & Kappeler (2021)

Breathers in the Benjamin–Ono equation





Breathers of the BO equation create no phase shifts.

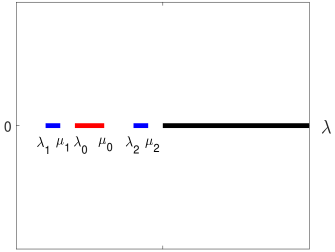
Chen & P., Wave Motion (2024)

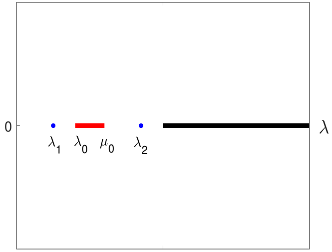
$$u = \frac{2(c_b + k\beta)\cosh\phi + [k(1+\beta^2+c_b^2\eta^2) + 2\beta c_b]\sinh\phi + 2c_b\cos(k\xi)}{(1+\beta^2+c_b^2\eta^2)\cosh\phi + 2\beta\sinh\phi + (1-\beta^2+c_b^2\eta^2)\cos(k\xi) + 2\beta c_b\eta\sin(k\xi}$$

where $\eta = x - c_b t - \eta_0$, $\beta = \frac{2c_b k}{(c_b - c)^2 - k^2}$, and

- either $c_b > c + k$ (for bright breathers)
- or $c_b < c - k$ (for dark breathers).

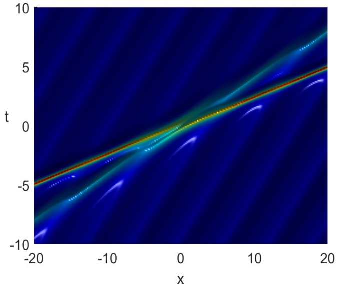
Breather solutions are obtained by degeneration of multi-periodic solutions in the long-wave limit, e.g. for two eigenvalues below.

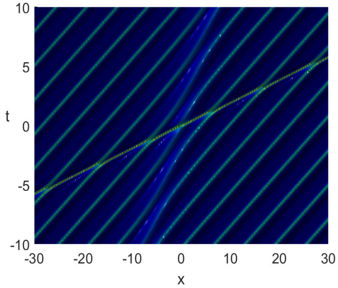


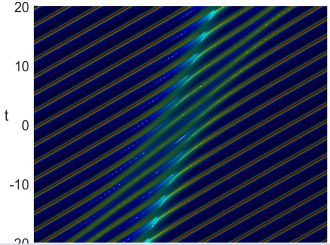


Breathers in the Benjamin–Ono equation

Bright-bright, bright-dark, and dark-dark breathers.







State-of-art of stability analysis of traveling waves

Recall that the TW solution of the KdV equation $u(x,t) = \phi(x - ct)$ is related to the Lax pair

$$\mathcal{L}(u)v = \lambda v, \quad \frac{\partial v}{\partial t} = \mathcal{M}(u)v,$$

for which we can separate the variables as $v(x,t) = w(x-ct)e^{\omega t}$.

State-of-art of stability analysis of traveling waves

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$$\mathcal{L}(u)v = \lambda v, \quad \frac{\partial v}{\partial t} = \mathcal{M}(u)v,$$

for which we can separate the variables as $v(x,t) = w(x - ct)e^{\omega t}$.

The “dispersion" relation $\omega = \omega(\lambda)$ can be found from the characteristic polynomial

$$\omega^2 + 16P(\lambda) = 0, \quad P(\lambda) = \lambda^3 + \frac{c}{2}\lambda^2 + \frac{c^2 - 4b}{16}\lambda - \frac{d + bc}{16},$$

which follows from the commutability of

$$\begin{aligned} w''(x) + \phi(x)w(x) + \lambda w(x) = 0, \\ -3\phi'(x)w(x) - 6\phi w'(x) - 4w'''(x) + cw'(x) = \omega w(x). \end{aligned}$$

State-of-art of stability analysis of traveling waves

Recall that the TW solution of the KdV equation $u(x,t) = \phi(x - ct)$ is related to the Lax pair

$$\mathcal{L}(u)v = \lambda v, \quad \frac{\partial v}{\partial t} = \mathcal{M}(u)v,$$

for which we can separate the variables as $v(x,t) = w(x - ct)e^{\omega t}$.

Linearized KdV equation at the TW with profile $\phi(x - ct)$ and perturbation $\mathfrak{u}(x - ct)e^{\Lambda t}$ can be solved by the squared eigenfunctions

$$\mathfrak{u}(x) = w(x)w'(x), \quad \Lambda = 2\omega = \pm 8i\sqrt{P(\lambda)},$$

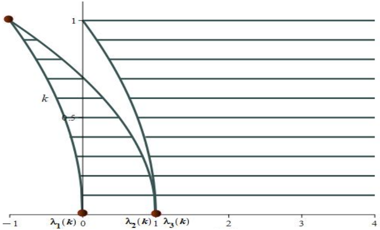
where $w(x)$ is the eigenfunction of the Lax system and

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3).$$

State-of-art of stability analysis of traveling waves

If $\phi(x+L) = \phi(x)$ is spatially periodic, then $w(x+L) = w(x)e^{i\kappa x}$ is required to be bounded which is only possible inside the spectral bands of the Lax spectrum:

$$\lambda \in \sigma_L := [\lambda_1, \lambda_2] \cup [\lambda_3, \infty).$$



Since $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) > 0$ for $\lambda \in \sigma_L$, then we have $\Lambda = \pm 8i\sqrt{P(\lambda)} \in i\mathbb{R}$, hence the TW is spectrally stable.

Section 2

Traveling waves and their spectral stability in the modified KdV equation

Traveling periodic waves in the modified KdV equation

The modified KDV equation is physically relevant in the same context of fluids. There are two meaningful cases

$$u_t \pm 6u^2 u_x + u_{xxx} = 0,$$

which are called focusing and defocusing, by the analogy to the NLS

$$i\psi_t + \psi_{xx} \pm 2|\psi|^2\psi = 0.$$

However, the NLS and mKdV equations are very different in the traveling wave solutions $u(x,t) = \phi(x - ct)$:

$$\phi''' \pm 6\phi^2\phi' - c\phi' = 0.$$

After integration(s), it yields

$$\phi'' \pm 2\phi^3 - c\phi = b \quad \Rightarrow \quad (\phi')^2 \pm \phi^4 - c\phi^2 = 2b\phi + d$$

Traveling periodic waves in the modified KdV equation

Since only one scaling transformation is available

$$\phi(x) = a\tilde{\phi}(ax), \quad c = a^2\tilde{c}, \quad b = a^3\tilde{b}, \quad d = a^4\tilde{d},$$

two parameters are independent.

If $b = 0$, the solutions are expressed in terms of Jacobi dn, cn, and sn functions like in NLS, e.g. in the defocusing case,

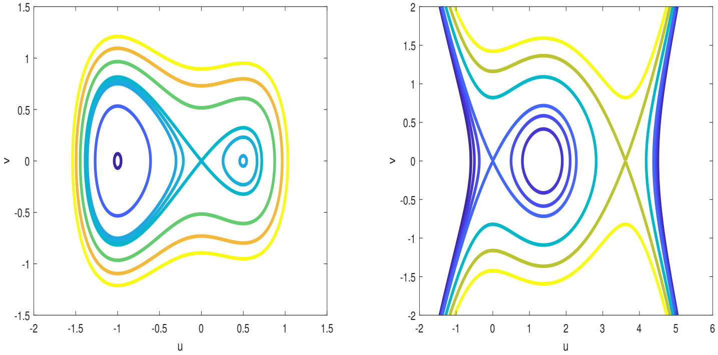
$$\phi(x) = k\,\mathrm{sn}(x,k), \quad c = 1 + k^2, \quad b = 0, \quad d = \frac{1}{2}k^2.$$

If $b \neq 0$, the solutions are expressed in elliptic functions,

$$\phi(x) = u_4 + \frac{(u_2 - u_4)(u_3 - u_4)}{(u_2 - u_4) - (u_2 - u_3)\mathrm{sn}^2(x,k)}, \quad k^2 = \frac{(u_1 - u_4)(u_2 - u_3)}{(u_1 - u_3)(u_2 - u_4)},$$

with a nontrivial dependence between (u_1, u_2, u_3, u_4) and (b, c, d).

Traveling periodic waves in the modified KdV equation



Focusing case (left): two families exist to generalize cn and dn.
Defocusing case (right): one family exists to generalize sn.

Algebraic characterization of traveling waves

The mKdV equation is a compatibility condition of the Lax system

$$\partial_x \varphi = \begin{pmatrix} i\zeta & u \\ u & -i\zeta \end{pmatrix} \varphi$$

and

$$\partial_t \varphi = \begin{pmatrix} 4i\zeta^3 + 2i\zeta u^2 & 4\zeta^2 u - 2i\zeta u_x + 2u^3 - u_{xx} \\ 4\zeta^2 u + 2i\zeta u_x + 2u^3 - u_{xx} & -4i\zeta^3 - 2i\zeta u^2 \end{pmatrix} \varphi.$$

If $u(x,t) = \phi(x - ct)$, then $\varphi(x,t) = \psi(x - ct) e^{\omega t}$ with ω found from the algebraic system

$$\begin{aligned} &\omega\psi - c \begin{pmatrix} i\zeta & \phi \\ \phi & -i\zeta \end{pmatrix} \psi \\ &= \begin{pmatrix} 4i\zeta^3 + 2i\zeta\phi^2 & 4\zeta^2\phi - 2i\zeta\phi' - c\phi - b \\ 4\zeta^2\phi + 2i\zeta\phi' - c\phi - b & -4i\zeta^3 - 2i\zeta\phi^2 \end{pmatrix} \psi. \end{aligned}$$

Algebraic characterization of traveling waves

The characteristic equation is

$$\omega^2 + 16P(\zeta) = 0, \quad P(\zeta) := \zeta^6 - \frac{c}{2}\zeta^2 + \frac{1}{16}(c^2 - 8d)\zeta^2 - \frac{b^2}{16},$$

which can be factorized as $P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)$.

Lax spectrum σ_L for traveling periodic waves is the Floquet spectrum

$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

where $\phi(x+L) = \phi(x)$ and $\psi(x+L) = e^{i\theta x}\psi(x)$ with $\theta \in \left[-\frac{\pi}{L}, \frac{\pi}{L}\right]$.

The polynomial $P(\zeta)$ is related to two important applications.

Algebraic characterization of traveling waves

The characteristic equation is

$$\omega^2 + 16P(\zeta) = 0, \quad P(\zeta) := \zeta^6 - \frac{c}{2}\zeta^2 + \frac{1}{16}(c^2 - 8d)\zeta^2 - \frac{b^2}{16},$$

which can be factorized as $P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)$.

Roots of $P(\zeta)$ correspond to either $\theta = 0$ or $\theta = \frac{\pi}{L}$ for which

$$\phi = p_1^2 + q_1^2 + p_2^2 + q_2^2,$$

where $\psi = (p_1, q_1)^T$ and $\psi = (p_2, q_2)^T$ are eigenvectors for ζ_1, ζ_2.

Cao & Geng, (1990) Chen & P., J. Nonlinear Science (2019)

Algebraic characterization of traveling waves

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which can be factorized as $P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)$.

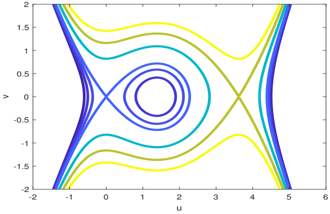
Linearized modified KdV equation at the traveling wave with profile $\phi(x - ct)$ and perturbation $v(x - ct)e^{\Lambda t}$ can be solved in terms of squared eigenfunctions:

$$v = p^2 - q^2, \qquad \Lambda = 2\omega = \pm 8i\sqrt{P(\zeta)},$$

where $\psi = (p, q)^T$ is an eigenvector for $\zeta \in \sigma_L$.

Deconinck & Nivala, (2011) Deconinck & Upsal (2021)

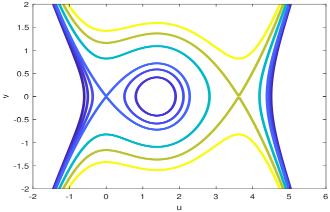
Spectral stability of traveling waves: defocusing case



$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

Spectral stability of traveling waves: defocusing case



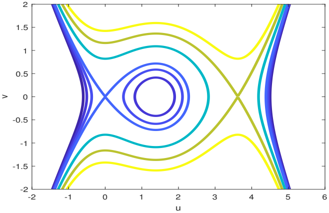
$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

Since the spectral problem is self-adjoint, the Lax spectrum $\sigma_L \subset \mathbb{R}$.

$P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)$ has real $0 \leq \zeta_3 \leq \zeta_2 \leq \zeta_1$.

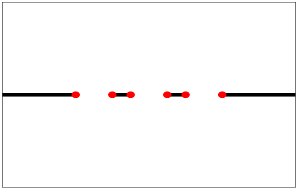
Spectral stability of traveling waves: defocusing case



$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

Lax spectrum σ_L is

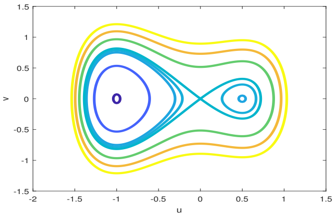


We have

$$P(\zeta) > 0 \qquad \zeta \in \sigma_L$$

hence
$\Lambda = \pm 8i\sqrt{P(\zeta)} \in i\mathbb{R}$ and
the periodic wave is modulationally stable.

Spectral stability of traveling waves: focusing case

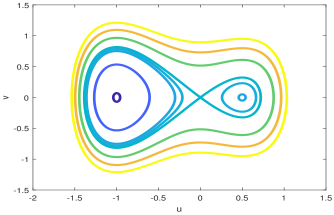


The spectral problem is

$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

Spectral stability of traveling waves: focusing case



The spectral problem is

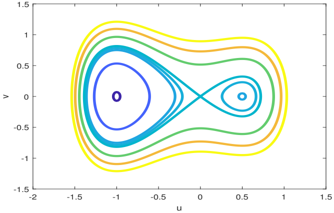
$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

The spectral problem is no longer self-adjoint and $\sigma_L \subset \mathbb{C}$.

$P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)$ has complex $\zeta_{1,2,3} \in \mathbb{C}$.

Spectral stability of traveling waves: focusing case

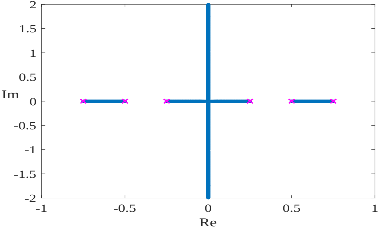


The spectral problem is

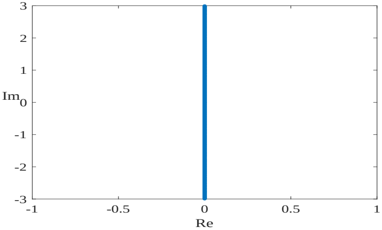
$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

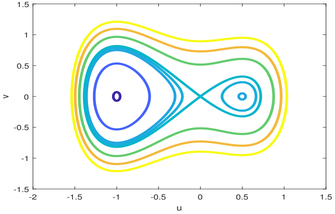
Lax spectrum σ_L for $\lambda = i\zeta$ is



Stability spectrum $\Lambda = \pm 8i\sqrt{P(\zeta)}$:



Spectral stability of traveling waves: focusing case

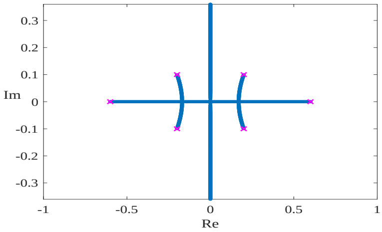


The spectral problem is

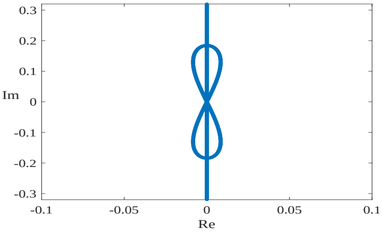
$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

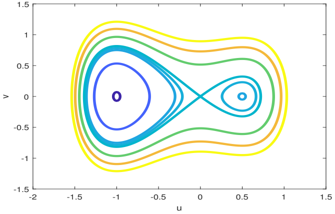
Lax spectrum σ_L for $\lambda = i\zeta$ is



Stability spectrum $\Lambda = \pm 8i\sqrt{P(\zeta)}$:



Spectral stability of traveling waves: focusing case



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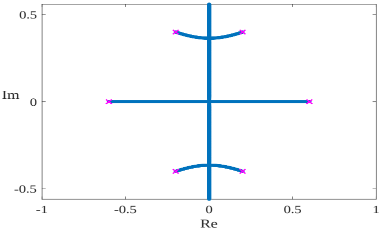
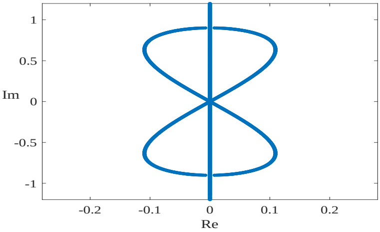
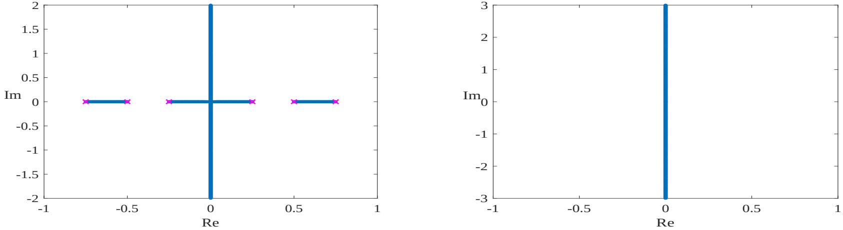


Stability spectrum $\Lambda = \pm 8i\sqrt{P(\zeta)}$:



Rogue waves in the focusing case

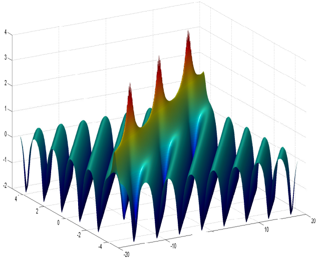


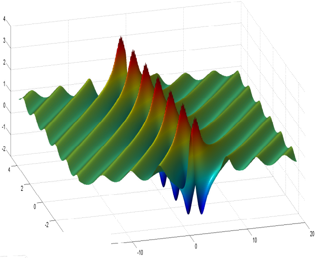
For each root of $P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_3^2)$, one can construct two solutions $\psi = (p, q)^T$ of the Lax system: one is bounded and the other one is linearly growing. With the Darboux transformation

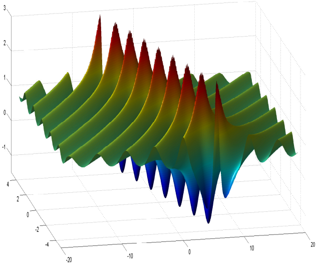
$$\hat{u} = u + \frac{4\lambda_1 pq}{p^2 + q^2},$$

the unbounded solution is used to construct rogue waves or algebraic solitons on the periodic background. Chen & P., J. Nonlinear Science (2019)

Rogue waves in the focusing case

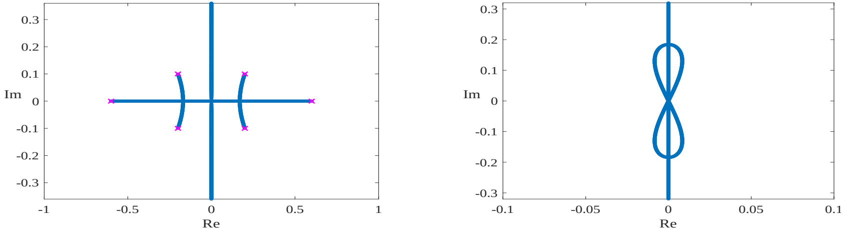






Three algebraic solitons exist because the background TW is stable.

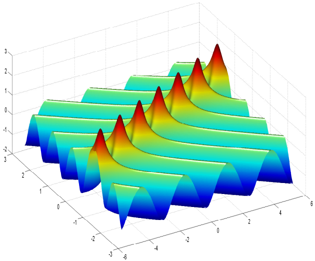
Rogue waves in the focusing case

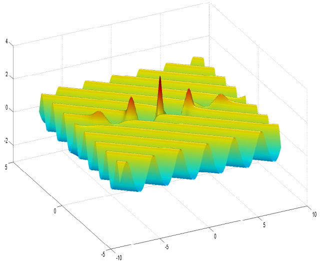


A two-fold Darboux transformation is needed for the complex-conjugate roots.

$$\tilde{u} = u + \frac{4(\lambda_1^2 - \lambda_2^2)\left[\lambda_1 p_1 q_1 (p_2^2 + q_2^2) - \lambda_2 p_2 q_2 (p_1^2 + q_1^2)\right]}{(\lambda_1^2 + \lambda_2^2)(p_1^2 + q_1^2)(p_2^2 + q_2^2) - 2\lambda_1\lambda_2[4 p_1 q_1 p_2 q_2 + (p_1^2 - q_1^2)(p_2^2 - q_2^2)]},$$

Rogue waves in the focusing case





Algebraic soliton exists for $\lambda_1 \in \mathbb{R}$ and the rogue wave exists for $\lambda_2 = \bar{\lambda}_3 \in \mathbb{C}$ in $P(\lambda) = (\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)(\lambda^2 - \lambda_3^2)$ because the background TW is unstable.

Section 3

Breathers in the defocusing modified KdV equation

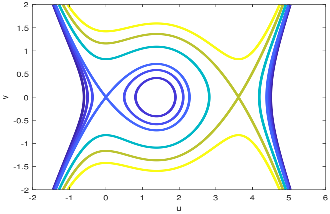
Recap of what was said before

Here we consider the defocusing MKDV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0,$$

with one family of traveling waves $u(x,t) = \phi(x+ct)$, $c > 0$ such that

$$(\phi')^2 - \phi^4 + c\phi^2 = 2b\phi + d$$



The spectral problem is:

$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x),$$

with $\phi(x+L) = \phi(x)$.

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The general periodic solution is expressed in elliptic functions,

$$\phi(x) = u_4 + \frac{(u_2 - u_4)(u_3 - u_4)}{(u_2 - u_4) - (u_2 - u_3)\mathrm{sn}^2(x,k)}, \quad k^2 = \frac{(u_1 - u_4)(u_2 - u_3)}{(u_1 - u_3)(u_2 - u_4)},$$

with a nontrivial dependence between (u_1, u_2, u_3, u_4) and (b, c, d).

Recap of what was said before

Here we consider the defocusing MKDV equation

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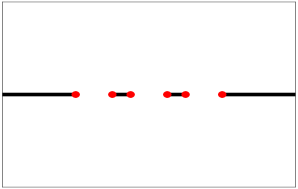
Lax spectrum σ_L is



We have

$$P(\zeta) > 0 \qquad \zeta \in \sigma_L$$

hence
$\Lambda = \pm 8i\sqrt{P(\zeta)} \in i\mathbb{R}$ and
the periodic wave is modulationally stable.

Recap of what was said before

Here we consider the defocusing MKDV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0,$$

with one family of traveling waves $u(x,t) = \phi(x+ct)$, $c > 0$ such that

$$(\phi')^2 - \phi^4 + c\phi^2 = 2b\phi + d$$

If $b = 0$, the solution is the same as in the NLS equation:

$$u(x,t) = \phi(x+ct), \quad \phi(x) = k\,\mathrm{sn}(x;k), \quad c = 1 + k^2, \quad k \in (0,1),$$

and the breathers have been constructed in the explicit form.

A. Mucalica & D. P., Lett. Math. Phys. (2024)

If $b \neq 0$, breathers have not been constructed.

L.K. Arruda & D. P., in progress (2024)

Construction of dark breathers for $b = 0$

There exists an exact solution of the Lax system for $\psi = (p, q)^T$:

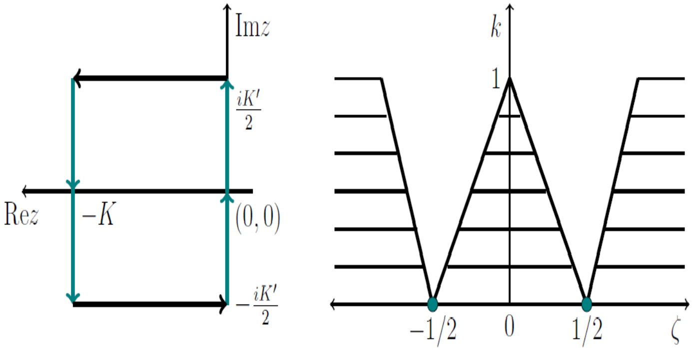
$$p = e^{sx+\omega t} e^{-\frac{i\pi x}{4K}} \frac{H(x-z)}{\Theta(x)\Theta(z)}, \qquad q = e^{sx+\omega t} e^{-\frac{i\pi x}{4K}} \frac{\Theta(x-z)}{\Theta(x)H(z)},$$

where $z \in \mathbb{C}$ is a smart choice of parameterization for which

$$\zeta = \frac{1}{2}\,\mathrm{dn}(z)\,\mathrm{dn}(iK' - z),$$
$$s = \frac{1}{2}Z(z) - \frac{1}{2}Z(iK' - z),$$
$$\omega = -4i\sqrt{(\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2)},$$

with $\zeta_3 = 0$ if $b = 0$. [D. A. Takahashi (2016)]

Construction of dark breathers for $b = 0$



Lax spectrum $\sigma_L = (-\infty, -\zeta_1] \cup [-\zeta_2, \zeta_2] \cup [\zeta_1, \infty)$.

Construction of dark breathers for $b = 0$

Dark breathers correspond to $\zeta \in (\zeta_2, \zeta_1)$ for which we can parameterize

$$z = \frac{iK'}{2} - \alpha, \quad \alpha \in (0, K)$$

and compute

$$\zeta = \frac{1+k}{2} \, \frac{1 - k\,\mathrm{sn}^2(\alpha)}{1 + k\,\mathrm{sn}^2(\alpha)} \in [\zeta_2, \zeta_1],$$

$$s = -Z(\alpha) - \frac{k\,\mathrm{sn}(\alpha)\,\mathrm{cn}(\alpha)\,\mathrm{dn}(\alpha)}{1 + k\,\mathrm{sn}^2(\alpha)} < 0,$$

$$\omega = -2k(1+k)^2 \frac{1 - k\,\mathrm{sn}^2(\alpha)}{[1 + k\,\mathrm{sn}^2(\alpha)]^3} \,\mathrm{sn}(\alpha)\,\mathrm{cn}(\alpha)\,\mathrm{dn}(\alpha) < 0.$$

Construction of dark breathers for $b = 0$

New solution is obtained with the Darboux transformation:

$$\hat{u} = u - \frac{4i\zeta pq}{p^2 - q^2},$$

where

$$p = \bar{q} = e^{-\eta} e^{-\frac{i\pi\xi}{4K}} \frac{H\left(\xi + \alpha - \frac{iK'}{2}\right)}{\Theta(\xi)\Theta\left(-\alpha + \frac{iK'}{2}\right)} + e^{\eta} e^{-\frac{i\pi\xi}{4K}} \frac{H\left(\xi - \alpha - \frac{iK'}{2}\right)}{\Theta(\xi)\Theta\left(\alpha + \frac{iK'}{2}\right)},$$

with $\xi := x + ct$ (periodic wave coordinate) and $\eta := -s(\xi + \xi_0) - \omega t$ (dark soliton coordinate). With (lots) of elliptic function relations (e.g. quarter-period translations of Jacobi theta's functions), one can obtain an exact solution for $\hat{u}(x,t)$.

However, the new solution $\hat{u}(x,t)$ is singular! A bounded solution is obtained after the transformation: $\tilde{u}(x,t) := \hat{u}(x + iK', t)$.

Construction of dark breathers for $b = 0$

If the traveling wave solution is

$$u(x,t) = k\,\mathrm{sn}(\xi;k) = \sqrt{k}\frac{H(\xi)}{\Theta(\xi)},$$

then the dark breather solution is

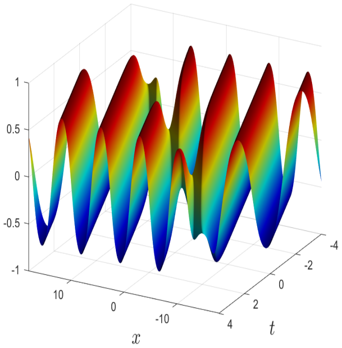
$$\tilde{u}(x,t) = \sqrt{k}\;\frac{H(\xi+2\alpha)e^{-2\eta} + H(\xi-2\alpha)e^{2\eta} + 2\beta H(\xi)}{\Theta(\xi+2\alpha)e^{-2\eta} + \Theta(\xi-2\alpha)e^{2\eta} + 2\gamma\Theta(\xi)},$$

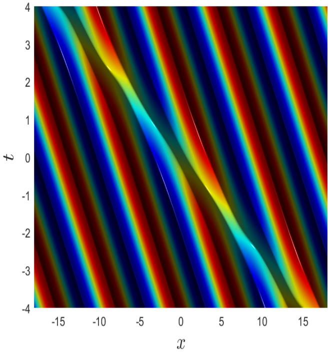
with some explicit $\beta, \gamma \in \mathbb{R}$ and $\xi = x + c_0 t$, $\eta = \kappa(x + ct + x_0)$.

For $k = 1$, this yields the two-soliton solution:

$$\tilde{u}(x,t) = \frac{\sinh(\xi+2\alpha)e^{-2\eta} + \sinh(\xi-2\alpha)e^{2\eta} + 2\sinh(\xi)(1-\sinh^2(2\alpha))\mathrm{sech}(2\alpha)}{\cosh(\xi+2\alpha)e^{-2\eta} + \cosh(\xi-2\alpha)e^{2\eta} + 2\cosh(\xi)\cosh(2\alpha)}.$$

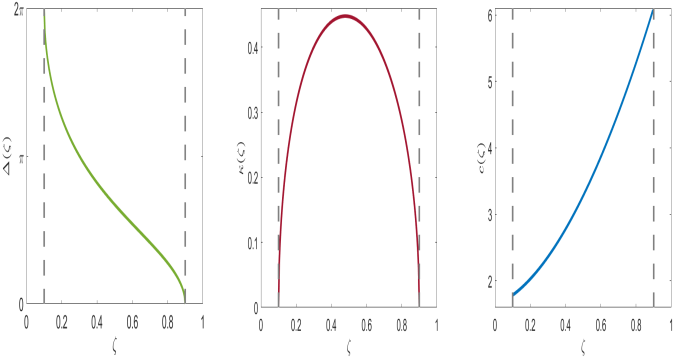
Construction of dark breathers for $b = 0$



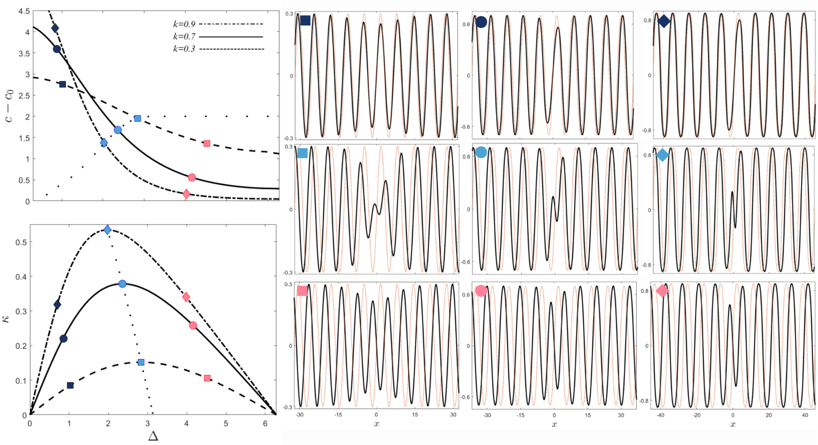


Construction of dark breathers for $b = 0$

Characteristics of the dark breather:



Construction of dark breathers for $b = 0$



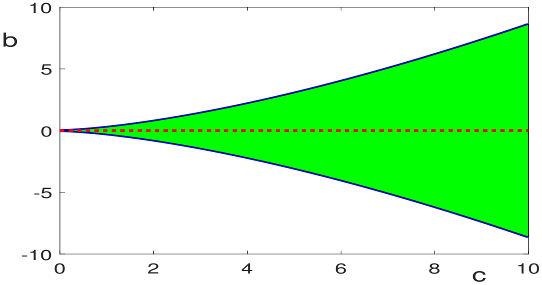
Towards construction of dark breathers for $b \neq 0$

The general periodic solution is expressed in elliptic functions,

$$\phi(x) = u_4 + \frac{(u_2 - u_4)(u_3 - u_4)}{(u_2 - u_4) - (u_2 - u_3)\mathrm{sn}^2(x,k)}, \quad k^2 = \frac{(u_1 - u_4)(u_2 - u_3)}{(u_1 - u_3)(u_2 - u_4)},$$

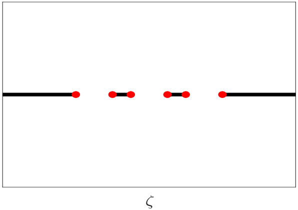
with a nontrivial dependence between (u_1, u_2, u_3, u_4) and (b, c, d).

Parameters (b, c) are defined inside the existence domain



Towards construction of dark breathers for $b \neq 0$

Lax spectrum σ_L is



$$P(\zeta) = (\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)(\zeta^2 - \zeta_2^2)$$

with $0 \leq \zeta_3 < \zeta_2 < \zeta_1$.

It was shown by A. Kamchatnov (1990,1999) that both the periodic solution and the domain can better be expressed by $\zeta_1, \zeta_2, \zeta_3$:

$$\phi(x) = \frac{2(\zeta_1 + \zeta_3)(\zeta_2 + \zeta_3)}{(\zeta_1 + \zeta_3) - (\zeta_1 - \zeta_2)\mathrm{sn}^2(x,k)} - \zeta_1 - \zeta_2 - \zeta_3, \quad k^2 = \frac{\zeta_1^2 - \zeta_2^2}{\zeta_1^2 - \zeta_3^2}.$$

Transformation $(\zeta_1, \zeta_2, \zeta_3) \mapsto (b, c, d)$ is an invertible diffeomorphism for $0 \leq \zeta_3 < \zeta_2 < \zeta_1$.

Towards construction of dark breathers for $b \neq 0$

Weierstrass' elliptic function $\wp(x)$ is related to $\mathrm{sn}^2(x,k)$, hence $\phi(x)$ is a linear fractional transformation of $\wp(x)$. Moreover, it was known for at least 100 years [N.I. Akhiezer, E.T. Whittaker–G.N. Watson] that

$$\phi^2(x) = \wp\left(x+v\right) + \wp\left(x-v\right) + \wp(2v).$$

and

$$\phi'(x) = \wp\left(x-v\right) - \wp\left(x+v\right),$$

where $\pm v$ are double poles of the elliptic functions $\phi^2(x)$ and $\phi'(x)$ inside the fundamental rectangle

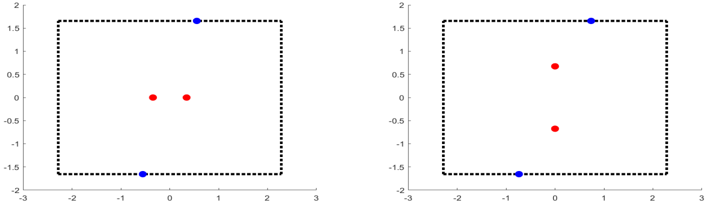
$$\mathcal{R} := \left\{z \in \mathbb{C} : \quad -K \leq \mathrm{Re}(z) \leq K, \quad -K' \leq \mathrm{Im}(z) \leq K'\right\},$$

Towards construction of dark breathers for $b \neq 0$

This brings up theory of elliptic functions:

- The number of zeros of $\phi(x)$ in $\mathcal{R}$ is equal to the number of poles of $\phi(x)$ in $\mathcal{R}$.
- The number of zeros and poles of $\wp(x)$ is equal to 2 in $\mathcal{R}$.
- Every elliptic function can be factorized as a quotient of the product of Jacobi's theta function $H(x)$.

We proved that poles of $\phi(x)$ are $\pm v = \pm(iK' + \alpha)$ with $\alpha \in (0, K)$.



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This yields the “optimal" representation of periodic solutions:

$$\phi(x) = C\frac{H(x-\beta)H(x+\beta)}{\Theta(x-\alpha)\Theta(x+\alpha)}, \quad C := (\zeta_1 - \zeta_2 - \zeta_3)\frac{\Theta^2(\alpha)}{H^2(\beta)},$$

where $\alpha \in \mathbb{R}$ and either $\beta \in \mathbb{R}$ or $\beta \in i\mathbb{R}$.

Towards construction of dark breathers for $b \neq 0$

This brings up theory of elliptic functions:

- The number of zeros of $\phi(x)$ in $\mathcal{R}$ is equal to the number of poles of $\phi(x)$ in $\mathcal{R}$.
- The number of zeros and poles of $\wp(x)$ is equal to 2 in $\mathcal{R}$.
- Every elliptic function can be factorized as a quotient of the product of Jacobi's theta function $H(x)$.

The symmetric case $b = 0$ (for which $\zeta_3 = 0$) is degenerate since it is not a particular limit of the general solution.

$$\phi(x) = k\,\mathrm{sn}(x,k) = \sqrt{k}\frac{H(x)}{\Theta(x)}.$$

Eigenfunctions of the Lax system for $b \neq 0$

We take the elliptic traveling wave solution

$$\phi(x) = C \frac{H(x-\beta)H(x+\beta)}{\Theta(x-\alpha)\Theta(x+\alpha)}, \quad C := (\zeta_1 - \zeta_2 - \zeta_3) \frac{\Theta^2(\alpha)}{H^2(\beta)}$$

and attempt to construct elliptic solutions of the Lax system:

$$\psi'(x) = \begin{pmatrix} i\zeta & \phi(x) \\ \phi(x) & -i\zeta \end{pmatrix} \psi(x).$$

Eigenfunctions of the Lax system for $b \neq 0$

For $\zeta \neq 0$, we can use ideas from [Belokolos–Bobenko-Enolskii–Its–Matveev, 1994] and write for $\psi = (p,q)^T$:

$$\frac{q}{p} = -i\frac{\zeta^3 + \frac{1}{2}\zeta(\phi^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2) - \sqrt{P(\zeta)}}{\zeta^2\phi - \frac{i}{2}\zeta\phi' - \zeta_1\zeta_2\zeta_3}$$
$$= -i\frac{\zeta^2\phi + \frac{i}{2}\zeta\phi' - \zeta_1\zeta_2\zeta_3}{\zeta^3 + \frac{1}{2}\zeta(\phi^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2) + \sqrt{P(\zeta)}}$$
$$= C\frac{H(x+z_1^*)H(x+z_2^*)}{H(x-z_1)H(x-z_2)},$$

where $\{\pm z_1, \pm z_2\}$ is the set of zeros of the second denominator and $\{\pm z_1^*, \pm z_2^*\}$ is the set of zeros of the first numerator such that

$$z_1 + z_2 + z_1^* + z_2^* = 0 \bmod (2K, 2K').$$

Eigenfunctions of the Lax system for $b \neq 0$

With further integration of

$$p'(x) = i\zeta p(x) + \phi(x) q(x),$$

we can obtain

$$p = e^{sx+\omega t} \frac{H(x - z_1) H(x - z_2)}{\Theta(x-\alpha)\Theta(x+\alpha)\Theta(\alpha - z_1)\Theta(\alpha - z_2)} e^{-\frac{i\pi}{2K}(z_1+z_2)},$$

$$q = -e^{sx+\omega t} \frac{H(x + z_1^*) H(x + z_2^*)}{\Theta(x-\alpha)\Theta(x+\alpha)\Theta(\alpha + z_1^*)\Theta(\alpha + z_2^*)} e^{\frac{i\pi}{2K}(z_1^*+z_2^*)},$$

with unique expression for s and $\omega = 4i\sqrt{P(\zeta)}$.

The main remaining question is
how to parameterize $\{z_1, z_2, z_1^*, z_2^*\}$ in terms of $\zeta \in \mathbb{R}$.

Eigenfunctions of the Lax system for $b \neq 0$

If $b = 0$ ($\zeta_3 = 0$), the elliptic theory gives for $\psi = (p, q)^T$:

$$\frac{q}{p} = -i \frac{\zeta^2 + \frac{1}{2}(\phi^2 - \zeta_1^2 - \zeta_2^2) - \sqrt{(\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)}}{\zeta\phi - \frac{i}{2}\phi'}$$
$$= -i \frac{\zeta\phi + \frac{i}{2}\phi'}{\zeta^2 + \frac{1}{2}(\phi^2 - \zeta_1^2 - \zeta_2^2) + \sqrt{(\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_2^2)}}$$
$$= C \frac{\Theta(x - z)}{H(x - z)},$$

where $\{\pm z\}$ is the set of zeros of the second denominator and $\{\pm(iK' - z)\}$ is the set of zeros of the first numerator and $z \in \mathbb{C}$ is related uniquely to $\zeta \in \mathbb{R}$ by

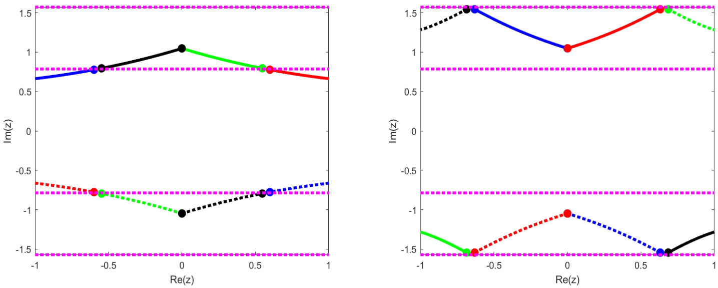
$$\frac{\phi'(z)}{\phi(z)} = \frac{\mathrm{cn}(z)\,\mathrm{dn}(z)}{\mathrm{sn}(z)} = -2i\zeta.$$

Eigenfunctions of the Lax system for $b \neq 0$

If $b \neq 0$, numerical results in the hyperbolic case $\zeta_2 = \zeta_3$ suggest that

$$z_1 = -\bar{z}_1^*, \qquad z_2 = -\bar{z}_2^*,$$

which still give three real parameters defined uniquely from $\zeta \in \mathbb{R}$.



Roots $\{\pm z_1, \pm z_2\}$ (blue,green) and $\{\pm z_1^*, \pm z_2^*\}$ (red,black) parameterized by ζ in (ζ_2, ζ_1) (left) and $(0, \zeta_3)$ (right).

Conclusion

The following questions were discussed in the context of elliptic traveling wave solutions of the basic integrable equations (KdV, mKdV, BO, NLS):

- ▷ How to connect the spectral stability problem with the Lax spectrum.
- ▷ How to use the elliptic theory in order to characterize the Lax spectrum and associated eigenfunctions.
- ▷ How to use the eigenfunctions to construct breathers (in the case of stability) and rogue waves (in the case of instability) on the traveling wave background.