

STUDY OF VORTICES IN TWO-DIMENSIONAL HARMONIC POTENTIALS

Math-790

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Introduction

- What is Bose-Einstein condensation (BEC)?

- The nonlinear evolution equation, called the Gross-Pitaevskii equation, models BEC in the mean-field approximation.

$$i\epsilon u_t + \epsilon^2(u_{xx} + u_{yy}) + (1 - x^2 - y^2 - |u|^2)u = 0.$$

where ϵ is a small parameter inversely proportional to chemical potential and $u(x, y, t) \in \mathbb{C}$ is the wave function.

- This project consists of following parts:
 - Study of vortex solutions near the bifurcation point.
 - Existence of a vortex solution.
 - Uniqueness of the positive vortex solution.
 - Numerical result.

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Local Bifurcation of Vortex Solutions

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- We define the Schrödinger operator \mathcal{H}_0 for a two-dimensional harmonic oscillator in the form:

$$\mathcal{H}_0 = -\epsilon^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + x^2 + y^2 - 1, \quad \epsilon > 0.$$

where the domain of \mathcal{H}_0 is:

$$\text{Dom}(\mathcal{H}_0) = \{f \in H^2(\mathbb{R}^2) : |x|^2 f \in L^2(\mathbb{R}^2)\}.$$

- The stationary Gross-Pitaevskii equation can be written in the form:

$$\mathcal{H}_0 u = -|u|^2 u.$$

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Local Bifurcation of Vortex Solutions

- To find the spectrum of \mathcal{H}_0 we write the eigenvalue equation:

$$\mathcal{H}_0 f = \lambda f, \quad f \in \text{Dom}(\mathcal{H}_0),$$

where λ stands for the eigenvalues and f for the corresponding eigenfunctions.

- Using the separation of variables to represent the wave function in the product form $f(x, y) = \varphi(x)\psi(y)$ and the properties of harmonic oscillators we can derive the eigenvalues of \mathcal{H}_0 as:

$$\sigma(\mathcal{H}_0) = \{ \lambda_{k,m}(\epsilon) = -1 + 2\epsilon(k + m + 1), (k, m) \in \mathbb{N}_0^2 \}.$$

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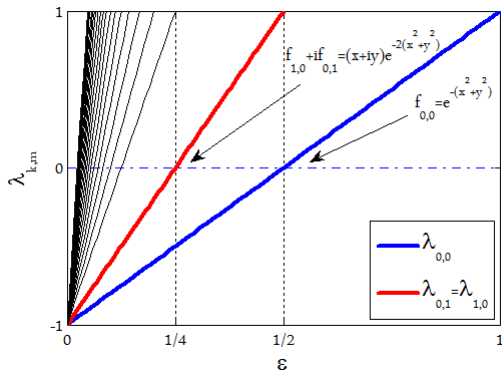


Figure 1: Eigenvalues $\lambda_{k,m}$ vs ϵ .

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- We use the method of Lyapunov-Schmidt Reduction to study the local bifurcation:

Theorem 1

Let $\mu = \frac{1}{16} - \epsilon^2$. There is $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$, there exist vortex solutions of the form,

$$u = \sqrt{128\mu}(x \pm iy)e^{-2(x^2+y^2)} + \mathcal{O}_{H^2(\mathbb{R}^2)}(\sqrt{\mu^3}),$$

in the stationary Gross-Pitaevski equation.

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in the stationary Gross-Pitaevski equation.

Existence of Vortex Solution

- We use calculus of variations to prove the existence of the vortex solution $u(x, y) = \phi(r)e^{i\theta}$ in polar coordinate (r, θ) on a truncated interval $[0, R] \ni r$.

- The associated energy functional is:

$$E_1(v) = \int_0^\infty \underbrace{\left[\epsilon^2 \left(\frac{dv}{dr} \right)^2 + \frac{\epsilon^2}{r^2} v^2 + (r^2 - 1)v^2 + \frac{1}{2}v^4 \right]}_{\text{energy density}} r dr.$$

- The stationary G-P equation is the Euler-Lagrange equation for $E_1(v)$:

$$\epsilon^2 \left(\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} - \frac{1}{r^2} \phi \right) + (1 - r^2 - \phi^2) \phi = 0.$$

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Theorem 2

For all $0 < \epsilon < \frac{1}{4}$, the energy functional $E_1(v)$, has a nonzero global minimizer ϕ in the energy space:

$$X_R = \{v \in H_r^1(0, R) : rv \in L_r^2(0, R)\}.$$

- Remark: To show that the energy functional is bounded we truncated \mathbb{R}_+ on a bounded interval $[0, R]$, subjected to the Dirichlet boundary condition $v|_{r=R} = 0$.

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Theorem 3

For any fixed $0 < \epsilon < \frac{1}{4}$, the positive vortex solution ϕ , is unique.

- Proof by ODE technique.
- Remark: the positivity of the solution is an extra assumption that we used.

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Numerical Approximation of The Vortex Solution.

- Shooting Method has been used to derive an approximation for the positive vortex solution for any fixed ϵ on the interval $[0, 1]$.

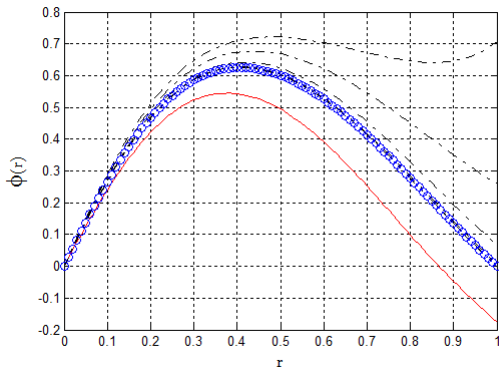


Figure 2: Vortex solution for $\epsilon = \frac{1}{6}$.

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- L-S Reduction suggest bifurcation of other (dipole) solutions for $\epsilon < \frac{1}{4}$. Persistence of the dipole solutions was not studied.
- Existence of vortex solution was proven on the truncated interval $[0, R]$. Can we prove the existence without the compactness assumption?
- We proved uniqueness of positive vortex solutions. Can we prove that positive vortex solutions exist?
- The shooting method was developed on the interval $[0, 1]$ but it has bad accuracy. What numerical methods can be used to improve the accuracy of numerical approximation?

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