

# Justification of a nonlinear Schrödinger model for polymers

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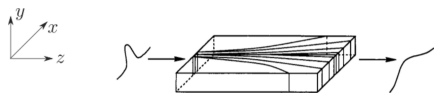
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May 3, 2012

## Photopolymers

- experience permanent change in refractive index under illumination due to triggered polymerization process;
- demonstrate wave guide trapping, mode evolution and other interesting nonlinear effects observed experimentally [Villafranca, Saravanamuttu, 2008];
- are modelled with Maxwell's and NLS equations [Monro et al, 2001].

# Model for pulse propagation



Wave-Maxwell system:

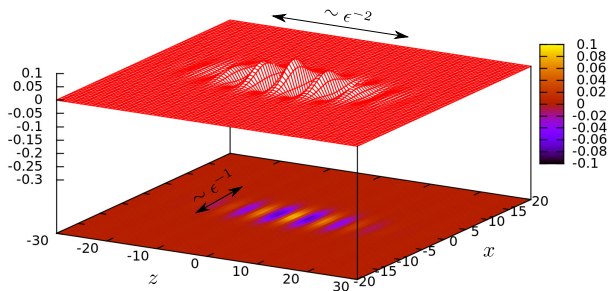
$$\begin{cases} \partial_x^2 E + \partial_z^2 E - (1 + m) \partial_t^2 E = 0, \\ \frac{\partial m}{\partial t} = E^2. \end{cases} \quad (x, z) \in \mathbb{R}^2, \quad t \in \mathbb{R}_+$$

Initial conditions:

$$\begin{aligned} E|_{t=0} &=: E_0 = \epsilon^{\frac{s+2}{2}} A_0 (\epsilon x, \epsilon^2 z) e^{i\omega_0 z} + c.c., \quad s \geq 2, \quad 0 < \epsilon \ll 1, \\ \partial_t E|_{t=0} &=: E_1 = -i\omega_0 \epsilon^{\frac{s+2}{2}} A_0 (\epsilon x, \epsilon^2 z) e^{i\omega_0 z} + c.c., \quad m|_{t=0} = 0. \end{aligned}$$

# Model for pulse propagation

Initial pulse:



NLS model:

$$\begin{cases} \partial_X^2 A + 2i\omega_0 (\partial_Z A + \partial_T A) + \omega_0^2 m_0 A = 0, \\ \partial_T m_0 = 2 |A|^2. \end{cases} \quad (X, Z) \in \mathbb{R}^2, \quad T \in \mathbb{R}_+$$

Initial conditions:  $A|_{T=0} =: A_0(X, Z)$ ,  $m_0|_{T=0} = 0$ .

## Reduction to a NLS model

Asymptotic expansion:

$$\begin{cases} E(x, z, t) = \epsilon^{\frac{s+2}{2}} \underbrace{\left( A(X, Z, T) e^{i\omega_0(z-t)} + c.c. \right)}_{=:(A)_{\omega_0}} + U(x, z, t), \\ m(x, z, t) = \epsilon^2 m_0(X, Z, T) + N(x, z, t), \end{cases}$$

where  $X := \epsilon x$ ,  $Z := \epsilon^2 z$ ,  $T := \epsilon^s t$ ,  $s \geq 2$ .

Choosing  $s = 2$ , leading order approximation yields the NLS model stated above; for  $s > 2$ , the NLS equation does not contain the  $\partial_T A$  term.

Our goal is to show that the residual terms  $U(x, z, t)$  and  $N(x, z, t)$  are indeed small !

# Main result

## Theorem

Given initial data  $A_0 \in H^8(\mathbb{R}^2)$ , let  $A, m_0$  be local solutions to the NLS system for  $T \in [0, T_0]$  with sufficiently small  $T_0 > 0$ . There exist  $\epsilon_0 > 0$  and a unique solution  $E, m$  of the wave-Maxwell system for  $t \in [0, T_0/\epsilon^2]$  such that, for any  $\epsilon \in (0, \epsilon_0)$ ,

$$\sup_{t \in [0, T_0/\epsilon^2]} \|E - \epsilon^2 (A)_{\omega_0}\|_{H^3(\mathbb{R}^2)} = \mathcal{O}(\epsilon^{5/2}),$$

$$\sup_{t \in [0, T_0/\epsilon^2]} \|m - \epsilon^2 m_0\|_{H^2(\mathbb{R}^2)} = \mathcal{O}(\epsilon^{5/2}).$$

# Idea of the proof

We obtain

- local well-posedness of the wave-Maxwell system (Kato's theory for symmetric hyperbolic systems);
- continuation arguments for the local solution (energy method);
- local well-posedness of the NLS system (Banach fixed-point theorem);
- residual equations in suitable for analysis form (near-identity transformations);
- control of error terms (a priori energy estimates).

## Wave-Maxwell system: local well-posedness

The wave-Maxwell system can be rewritten in the form

$$\begin{cases} \partial_t \mathbf{v} + A_1(\mathbf{v}) \partial_x \mathbf{v} + A_2(\mathbf{v}) \partial_z \mathbf{v} = \mathbf{f}(\mathbf{v}), \\ \mathbf{v}|_{t=0} = \mathbf{v}_0 \end{cases}$$

where  $\mathbf{v} := \left( \partial_t E, \frac{\partial_x E}{(1+m)^{1/2}}, \frac{\partial_z E}{(1+m)^{1/2}}, E, \partial_x m, \partial_z m, m \right)^T$ ,

$\mathbf{v}_0 := (E_1, \partial_x E_0, \partial_z E_0, E_0, 0, 0, 0)^T$ , and  $A_1, A_2$  are symmetric matrices.

Then, Kato's theory for symmetric quasi-linear hyperbolic systems gives, for some  $t_0 > 0$ ,

$$\mathbf{v} \in C([0, t_0], H^s(\mathbb{R}^2)) \cap C^1([0, t_0], H^{s-1}(\mathbb{R}^2))$$

providing  $\mathbf{v}_0 \in H^s(\mathbb{R}^2)$ .



# Wave-Maxwell system: local well-posedness

## Proposition

*For any integer  $s \geq 3$ , the unique local solution of the wave-Maxwell system exists in the space*

$$E \in C([0, t_0], H^{s+1}(\mathbb{R}^2)) \cap C^1([0, t_0], H^s(\mathbb{R}^2)) \cap C^2([0, t_0], H^{s-1}(\mathbb{R}^2))$$

$$m \in C^1([0, t_0], H^{s+1}(\mathbb{R}^2)) \cap C^2([0, t_0], H^s(\mathbb{R}^2)) \cap C^3([0, t_0], H^{s-1}(\mathbb{R}^2))$$

and depends continuously on the initial data  $E_0, E_1 \in H^s(\mathbb{R}^2)$ .

# Wave-Maxwell system: local well-posedness

## Theorem (continuation of local solution)

*Local solution of the wave-Maxwell system does not blow up in  $H^4$ -norm as  $t \rightarrow t_0$  if*

$$\sup_{t \in [0, t_0]} \left( \|E\|_{L^\infty(\mathbb{R}^2)} + \|\partial_t E\|_{L^\infty(\mathbb{R}^2)} + \|\nabla E\|_{L^\infty(\mathbb{R}^2)} \right) < \infty,$$

*and the initial data norms  $\|E_0\|_{H^3(\mathbb{R}^2)}$  and  $\|E_1\|_{H^2(\mathbb{R}^2)}$  are small enough.*

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We prove this by energy method. The derivatives  $\partial^\alpha E$ ,  $|\alpha| \leq 4$ , can be controlled in  $L^2$ -norm by constructing four energy functionals which are consequently shown to be bounded on times  $t \in [0, t_0]$ .

# NLS system: local well-posedness

## Theorem

For any integer  $s \geq 2$  and  $\delta > 2 \sup_{T \in \mathbb{R}_+} \|A_0\|_{H^s(\mathbb{R}^2)}$ , there exist a positive  $T_0$  and a unique solution

$$A \in C([0, T_0], H^s(\mathbb{R}^2)) \cap C^1([0, T_0], H^{s-2}(\mathbb{R}^2))$$

to the NLS system with  $A|_{T=0} = A_0$  and  $\sup_{T \in [0, T_0]} \|A\|_{H^s(\mathbb{R}^2)} \leq \delta$ .

# NLS system: local well-posedness

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Proof is application of the Banach fixed-point theorem to

$$A(X, Z, T) = S_T(X) \star A_0(X, Z - T) + i \int_0^T S_{T-\tau}(X) \star [m_0(X, Z - T + \tau, \tau) A(X, Z - T + \tau, \tau)] d\tau,$$

$$\text{where } S_T(X) := \frac{1}{\sqrt{4\pi T}} e^{-\frac{i\pi}{4}} e^{\frac{iX^2}{4T}}.$$

## Residual equations

Recall

$$\begin{cases} E(x, z, t) = \epsilon^2 (A)_{\omega_0} + U(x, z, t), \\ m(x, z, t) = \epsilon^2 m_0 + N(x, z, t). \end{cases}$$

Residual terms solve the equations

$$\begin{cases} \partial_x^2 U + \partial_z^2 U - (1 + \epsilon^2 m_0 + N) \partial_t^2 U = -\epsilon^2 \left( R_2^{(U)} \right)_{\omega_0} N - \epsilon^6 \left( R_6^{(U)} \right)_{\omega_0}, \\ \partial_t N = \epsilon^4 (A^2)_{2\omega_0} + 2\epsilon^2 (A)_{\omega_0} U + U^2 \end{cases}$$

with

$$U|_{t=0} = \partial_t U|_{t=0} = 0, \quad N|_{t=0} = 0,$$

where  $R_2^{(U)} := \omega_0^2 A + 2i\omega_0 \epsilon^2 \partial_T A - \epsilon^4 \partial_T^2 A,$

$$R_6^{(U)} := \partial_Z^2 A - (1 + \epsilon^2 m_0) \partial_T^2 A + 2i\omega_0 m_0 \partial_T A.$$

# Residual equations

The normal form transformation

$$V = U - \epsilon^4 (B)_{\omega_0} - \epsilon^4 (D)_{3\omega_0},$$

$$M = N - \epsilon^4 N_0 + 2\epsilon^2 \left( \frac{A}{i\omega_0} \right)_{\omega_0} V + \epsilon^4 \left( \frac{A^2}{2i\omega_0} \right)_{2\omega_0} + \epsilon^6 R_6^{(M)}$$

allows us to work with

$$\begin{cases} \partial_x^2 V + \partial_z^2 V - (1 + \epsilon^2 m_0 + N) \partial_t^2 V = 2i\omega_0 \epsilon^4 |A|^2 V - \epsilon^2 \omega_0^2 (A)_{\omega_0} M \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad - \epsilon^8 R_8^{(V)}, \\ \partial_t M = \epsilon^8 R_8^{(M)} + 2\epsilon^4 R_4^{(M)} V + 2\epsilon^2 \left( \frac{A}{i\omega_0} \right)_{\omega_0} \partial_t V + V^2. \end{cases}$$

## Control of error terms

The following quantities can be controlled, for  $t \in [0, T_0/\epsilon^2]$ ,

$$\mathcal{H}_1 := \frac{1}{2} \int_{\mathbb{R}^2} [(1 + \epsilon^2 m_0 + N) V_t^2 + V_x^2 + V_z^2] dx dz,$$

$$\begin{aligned} \mathcal{H}_2 := & \frac{1}{2} \int_{\mathbb{R}^2} [(1 + \epsilon^2 m_0 + N) V_{tt}^2 + (2 + \epsilon^2 m_0 + N) (V_{xt}^2 + V_{zt}^2) \\ & + V_{xx}^2 + V_{zz}^2 + 2V_{xz}^2] dx dz, \end{aligned}$$

$$\begin{aligned} \mathcal{H}_3 := & \frac{1}{2} \int_{\mathbb{R}^2} \left[ (1 + \epsilon^2 m_0 + N) (V_{xxt}^2 + V_{zzt}^2 + V_{ttt}^2) + \frac{1}{2} (V_{xxx}^2 + V_{zzz}^2) \right. \\ & \left. + V_{xxz}^2 + V_{xzz}^2 + \frac{1}{2} (V_{xxx} - 2N_x V_{tt})^2 + \frac{1}{2} (V_{zzz} - 2N_z V_{tt})^2 \right] dx dz. \end{aligned}$$



## Control of error terms

For example, multiplying the first equation by  $\partial_t U$  and integrating in  $x, z$  over  $\mathbb{R}^2$ :

$$\begin{aligned} \frac{d\mathcal{H}_1}{dt} &\leq 2\epsilon^4 \|A\|_{L^\infty}^2 \mathcal{H}_1 + \|N_t\|_{L^\infty} \mathcal{H}_1 + 2\sqrt{2}\epsilon^2 \omega_0^2 \|A\|_{L^\infty} \|M\|_{L^2} \mathcal{H}_1^{1/2} \\ &\quad + 2\sqrt{2}\epsilon^4 \omega_0 \|A\|_{L^\infty}^2 \|V\|_{L^2} \mathcal{H}_1^{1/2} + \sqrt{2}\epsilon^{13/2} \left\| R_8^{(V)} \right\|_{L^2_{x,z}} \mathcal{H}_1^{1/2}. \end{aligned}$$

Using the second equation and the estimates

$$\|\cdot\|_{L^2} \leq T_0 \epsilon^{-2} \sup_{T \in [0, T_0]} \|\partial_t(\cdot)\|_{L^2}, \quad \text{for } t \in [0, T_0/\epsilon^2],$$

$$\frac{d\left(\mathcal{H}_1^{1/2}\right)}{dt} \leq l_1 \mathcal{H}_1^{1/2} + J_1 \quad \Rightarrow \quad \mathcal{H}_1^{1/2}(t) \leq T_0 \epsilon^{-2} J_1 e^{l_1 T_0 \epsilon^{-2}}.$$

## Control of error terms

Eventually, we obtain the bounds

$$\sup_{t \in [0, T_0/\epsilon^2]} \|V\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right), \quad \sup_{t \in [0, T_0/\epsilon^2]} \|\nabla V\|_{L^2} = \mathcal{O}\left(\epsilon^{9/2}\right),$$

$$\sup_{t \in [0, T_0/\epsilon^2]} \|\Delta V\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right), \quad \sup_{t \in [0, T_0/\epsilon^2]} \|\nabla \Delta V\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right),$$

$$\sup_{t \in [0, T_0/\epsilon^2]} \|M\|_{L^2} = \mathcal{O}\left(\epsilon^{9/2}\right), \quad \sup_{t \in [0, T_0/\epsilon^2]} \|\nabla M\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right),$$

$$\sup_{t \in [0, T_0/\epsilon^2]} \|\Delta M\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right).$$

## Numerical illustrations in 1D

1D set-up:

$$\begin{cases} \partial_Z^2 U - (1 + \epsilon^2 m_0 + N) \partial_t^2 U = -\epsilon^2 \left( R_2^{(U)} \right)_{\omega_0} N - \epsilon^6 \left( R_6^{(U)} \right)_{\omega_0}, \\ \partial_t N = \epsilon^4 \left( A^2 \right)_{2\omega_0} + 2\epsilon^2 \left( A \right)_{\omega_0} U + U^2. \end{cases}$$

The choice of initial data  $A_0(Z) = \operatorname{sech}(Z)$  leads to

$$\begin{aligned} A(Z, T) &= \operatorname{sech}(Z - T) \exp \left[ i\omega_0 \left( \log \left( \frac{\cosh Z}{\cosh(Z - T)} \right) - T \tanh(Z - T) \right) \right], \\ m_0(Z, T) &= 2(\tanh Z - \tanh(Z - T)). \end{aligned}$$

# Numerical illustrations in 1D

Rewriting

$$\begin{cases} \partial_t^2 U = \frac{1}{a+N} \partial_z^2 U - \frac{bN+c}{a+N}, \\ \partial_t N = U^2 + dU + e, \end{cases}$$

we use explicit 2-step Adams-Bashforth and Crank-Nicholson methods

$$N_i^{(j+1)} = N_i^{(j)} + \frac{3}{2} \Delta t \left[ (U_i^{(j)})^2 + d_i^{(j)} U_i^{(j)} + e_i^{(j)} \right] - \frac{1}{2} \Delta t \left[ (U_i^{(j-1)})^2 + d_i^{(j-1)} U_i^{(j-1)} + e_i^{(j-1)} \right],$$

$$\begin{aligned} \frac{U_i^{(j+1)} - 2U_i^{(j)} + U_i^{(j-1)}}{(\Delta t)^2} &= \frac{1}{2} \left[ \frac{1}{a_i^{(j+1)} + N_i^{(j+1)}} \left( \frac{U_{i+1}^{(j+1)} - 2U_i^{(j+1)} + U_{i-1}^{(j+1)}}{(\Delta z)^2} - b_i^{(j+1)} N_i^{(j+1)} - c_i^{(j+1)} \right) \right. \\ &\quad \left. + \frac{1}{a_i^{(j-1)} + N_i^{(j-1)}} \left( \frac{U_{i+1}^{(j-1)} - 2U_i^{(j-1)} + U_{i-1}^{(j-1)}}{(\Delta z)^2} - b_i^{(j-1)} N_i^{(j-1)} - c_i^{(j-1)} \right) \right]. \end{aligned}$$

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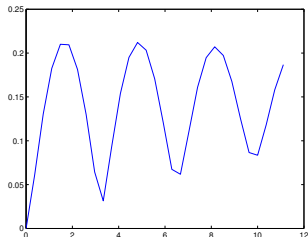
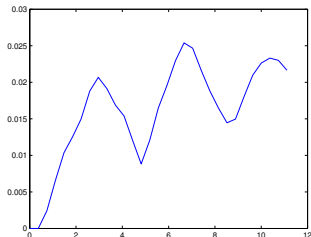


Figure:  $\|U\|_{H^3(\mathbb{R})}$  versus time  $t$

Figure:  $\|N\|_{H^2(\mathbb{R})}$  versus time  $t$

- Solutions for  $\epsilon = 0.3$  animated:  $U$ ,  $N$ .
- Estimates:

$$\sup_{t \in [0,1]} \|U\|_{H^3(\mathbb{R})} = \mathcal{O}(\epsilon^{\hat{\alpha}}), \quad \sup_{t \in [0,1]} \|N\|_{H^2(\mathbb{R})} = \mathcal{O}(\epsilon^{\hat{\beta}}),$$

where  $\hat{\alpha} = 4.9972$ ,  $\hat{\beta} = 3.0028$ .

The End

Thank You for the attention !!!