Matrix multiplication

Let A be a $1 \times n$ matrix and B an $n \times 1$ matrix. The **dot (inner) product** of A and B is a scalar number, obtained by multiplication of each entry of A with the corresponding entry of B and addition of the results:

$$A \cdot B = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} = \sum_{k=1}^{n} a_{1k}b_{k1}.$$

The **outer product** of B and A is a $n \times n$ matrix, obtained by multiplication of each entry of B with the corresponding entry of A:

$$B \cdot A = \begin{pmatrix} b_{11}a_{11} & b_{11}a_{12} & \dots & b_{11}a_{1n} \\ b_{21}a_{11} & b_{21}a_{12} & \dots & b_{21}a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ b_{n1}a_{11} & b_{n1}a_{12} & \dots & b_{n1}a_{1n} \end{pmatrix}$$

Example

$$A = \begin{bmatrix} 2 & -1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$$

Let A be an $m \times n$ matrix and B an $k \times p$ matrix. The **product** of A and B is defined only if $n = k$. The product of A and B is a $m \times p$ matrix, such that the (i, j)-entry is the dot product between row i of A and column j of B.

If $A = [a_{ik}]_{m\times n}$ and $B = [b_{kj}]_{n\times p}$, then $A \cdot B = [c_{ij}]_{m\times p}$ where

$$c_{ij} = \begin{bmatrix} a_{i1} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + \cdots + a_{in} b_{nj}.$$

Row-by-column multiplication:

$$(m \times n) \cdot (n \times p) = (m \times p)$$

Example

$$A = \begin{bmatrix} 5 & 2 \\ 2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$$

- The product $A \cdot B$ of two matrices A and B is defined if the number of *columns* of A equals to the number of *rows* of B.
- The resulting matrix $A \cdot B$ has as many rows as A and as many columns as B.
- Any square matrix with all entries on its diagonal equal to 1 and all other entries equal to 0 is called an **identity matrix**. The $n \times n$ identity matrix is usually denoted by I_n.
- If A is a $m \times n$ matrix, then

$$A \cdot I_n = A, \qquad I_m \cdot A = A$$

Theorem: Let A, B and C be square matrices of the same size and let k be a scalar. Then:

1. $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ (associativity)
2. $A \cdot (B + C) = A \cdot B + A \cdot C$ (distributivity I)
3. $(A + B) \cdot C = A \cdot C + B \cdot C$ (distributivity II)
4. $k(A \cdot B) = (kA) \cdot B = A \cdot (kB)$ (scalar multiplication)
5. $(A \cdot B)^T = B^T \cdot A^T$ (transpose)

Warning:

$$A \cdot B \neq B \cdot A$$

Example

Let A and B be symmetric matrices. Show that $A \cdot B$ is a symmetric matrix if and only if $A \cdot B = B \cdot A$.

If $A \cdot B = B \cdot A$, then A and B are called **commuting** matrices.

Example

Let A, B, C be square matrices and $A \cdot C = C \cdot A$ and $B \cdot C = C \cdot B$. Show that $A \cdot B$ computes with C, such that

$$(A \cdot B) \cdot C = C \cdot (A \cdot B).$$

Example

Let $a \neq 0$ and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Compute LU factorization of A, where

$$L = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} y & z \\ 0 & w \end{pmatrix}$$