Matrix form of a system of linear equations

Consider the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

Let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ldots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

The system is rewritten in the **matrix form**:

$$AX = B$$

Example

$$\begin{aligned} 5x_1 + 2x_2 - x_3 &= 6 \\ -2x_1 - x_2 + 4x_3 &= 3 \end{aligned}$$

Notations: A is the coefficient matrix, B is the constant matrix (vector), and X is the matrix (vector) of variables.

Block structure of a matrix

- Matrix is a column of rows

$$A = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}$$

where each R_k is a $1 \times n$ matrix (row-vector)

- Matrix is a row of columns

$$A = \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}$$

where each C_k is a $m \times 1$ matrix (column-vector)

The matrix form of the linear system can be rewritten in two equivalent forms:

$$A \cdot X = \begin{bmatrix} R_1 X \\ \vdots \\ R_m X \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

and

$$A \cdot X = \begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ \vdots \\ X_n X \end{bmatrix} = X_1 C_1 + \cdots X_n C_n = B$$

Principle of linear superposition

Let A be the $m \times n$ coefficient matrix, B be the $m \times 1$ constant vector, and X be the $n \times 1$ vector of variables.

- If the homogeneous system $A \cdot X = O$ has two particular solutions X_1 and X_2, then it also has the solution:

$$X = c_1 X_1 + c_2 X_2,$$

  where c_1 and c_2 are arbitrary constants.

- If X_1 is a particular solution of the inhomogeneous system $A \cdot X = B$, then the inhomogeneous system has the solution:

$$X = X_1 + X_0,$$

  where X_0 is a solution of the homogeneous system $A \cdot X = O$.

Example

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

Particular solutions X_1, X_2 etc of the homogeneous system $A \cdot X = O$ are called **basic (fundamental)** solutions. The solution in the form $X = c_1X_1 + c_2X_2$ is called a **linear combination** of basic solutions.

Theorem: Suppose the homogeneous system $AX = 0$ has n variables and the coefficient matrix A has rank r. Then

1. The homogeneous system has exactly $n - r$ basic solutions.
2. Every solution of the system is a linear combination of the basic solutions

$$X = c_1X_1 + c_2X_2 + ... + c_{n-r}X_{n-r}$$

Example

$$
\begin{aligned}
2x_2 + 3x_3 - 2x_4 &= 0 \\
2x_1 + x_2 - 4x_3 + 3x_4 &= 0 \\
2x_1 + 3x_2 + 2x_3 - x_4 &= 0
\end{aligned}
$$