Matrix inverses

Let A and B be *square* $n \times n$ matrices. If

$$A \cdot B = B \cdot A = I_n,$$

then the matrix B is called an **inverse** of A ($B = A^{-1}$) and the matrix A is called **invertible**.

Examples

$$n = 1: \qquad A = a, \qquad A^{-1} = \frac{1}{a}$$

$$n = 2: \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $a \neq 0$ ($n = 1$) and $ad - bc \neq 0$ ($n = 2$).

Theorem:

- If A is invertible, then A^{-1} is uniquely defined.
- Only square matrices may have inverses.

If the inverse of A does not exist, the matrix A is called **singular**.

Example

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

The method to find the inverse A^{-1}

1. Start with the given $n \times n$ matrix A and the $n \times n$ identity matrix I_n and construct the augmented matrix

$$[A|I_n]$$

2. Use elementary row operations and reduce the augmented matrix to the reduced row-echelon form.

3. If the resulting matrix is of the form $[I_n|B]$, then $B = A^{-1}$. Otherwise, no A^{-1} exists.

Example

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$$

Consider the linear inhomogeneous system $AX = B$, where A is of size $m \times n$ (X is of size $n \times 1$ and B is of size $m \times 1$).

- If $n > m$, the system is under-determined and some variables are arbitrary parameters
- If $n < m$, the system is over-determined. It has no solutions, unless at least $(m - n)$ equations are redundant
- If $n = m$ and A is invertible, there exists a unique solution:

$$X = A^{-1}B.$$

Example

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ or } B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Theorem: Let A be an $n \times n$ matrix. The following statement are equivalent (if and only if):

1. A is invertible.
2. There exists an $n \times n$ matrix A^{-1} such that $A^{-1} \cdot A = A \cdot A^{-1} = I_n$.
3. The homogeneous system $AX = 0$ has only the trivial solution $X = O$.
4. The inhomogeneous system $AX = B$ has a unique solution $X = A^{-1}B$.
5. the reduced row-echelon form of A is I_n

Properties of matrix inverses

Let A and B be invertible matrices and k be a non-zero number.

1. $(A^{-1})^{-1} = A$
2. $(A^{-1})^T = (A^T)^{-1}$
3. $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$
4. $(A^k)^{-1} = (A^{-1})^k$
5. $(kA)^{-1} = \frac{1}{k} A^{-1}$

Example

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

Compute A^2, $(A^{-1})^2$ and $(A^2)^{-1}$