Elementary matrices

A square matrix is called **elementary** if it is obtained from the identity matrix by a single elementary row operation.

Three types of elementary row operations:

- interchange two rows

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- multiply a row by a non-zero number k

$$\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$$

- Add a multiple of a row to a different row

$$\begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem: Let A be a $m \times n$ matrix and E be a $m \times m$ elementary matrix. Then, $E \cdot A$ is a $m \times n$ matrix, which is obtained from A by the same elementary row operation as in E.

Example

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 0 & -2 & 7 \end{bmatrix}$$

Therefore, a sequence of elementary row operations can be encoded in elementary matrices E and it can be performed on A by means of matrix multiplications.

Theorem: Every elementary matrix E is invertible and the inverse matrix E^{-1} performs the opposite elementary row operation to that of E.

<u>Example</u>

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem: Let A be a $m \times n$ matrix. There exists an invertible $m \times m$ matrix U such that $U \cdot A$ is a reduced row-echelon form of A and U is a product of all elementary matrices that encode a sequence of elementary row operations in the Gaussian elimination algorithm.

Example

$$A = \begin{bmatrix} 5 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix}$$

Theorem: A square matrix is invertible if and only if it is a product of elementary matrices.

<u>Example</u>

$$A = \begin{bmatrix} 5 & 0 \\ 3 & 2 \end{bmatrix}$$