Properties of the determinant Let $A$ be a square $n \times n$ matrix.

1. If $A$ has a row or column consisting entirely of zeros, then $\det(A) = 0$.
2. If two distinct rows (or two distinct columns) of $A$ are interchanged, the determinant changes the sign to $-\det(A)$.
3. If a row (or column) of $A$ is multiplied by a number $k$, the determinant is also multiplied by $k$ to $k \det(A)$.
4. If two rows (or two columns) of $A$ are identical, the determinant is zero.
5. If a multiple of one row is added to a different row (or if a multiple of one column is added to a different column), the determinant remains the same, that is $\det(A)$.
6. If $k$ is a non-zero number, then

$$\det(kA) = k^n \det(A).$$

Example

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}$$

$$(1)A = \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}, \qquad (2)A = \begin{bmatrix} 2 & 2 \\ 3 & -1 \end{bmatrix}, \qquad (3)A = \begin{bmatrix} 9 & -3 \\ 2 & 2 \end{bmatrix}$$

$$(4)A = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}, \qquad (5)A = \begin{bmatrix} 3 & -1 \\ 5 & 1 \end{bmatrix}, \qquad (6)A = \begin{bmatrix} 6 & -2 \\ 4 & 4 \end{bmatrix}$$

Properties of the determinant can be used to create zeros by using elementary row (or column) operations. These operations simplify computations of the determinants.

Example

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & -2 \\ 3 & 1 & 1 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{bmatrix}$$

- Let $A$ be an upper triangular matrix (or lower triangular) matrix, i.e. all entries below the main diagonal (above the main diagonal) are zeros. Then, the determinant is a product of all entries of the main diagonal of $A$:

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

- If a matrix can be sub-divided into blocks, and blocks above (or below) the main diagonal are zeros, then the determinant of the original matrix is the product of determinants of the matrices along the main diagonal.

  If $A = \begin{bmatrix} A_1 & X \\ O & A_2 \end{bmatrix}$ or $A = \begin{bmatrix} A_1 & O \\ X & A_2 \end{bmatrix}$, then

$$\det(A) = \det(A_1)\det(A_2).$$

  Here $A_1$ and $A_2$ are square matrices, $O$ is the matrix of zeros, and $X$ is a matrix of suitable size.

Example

$$A = \begin{bmatrix} 1 & 3 & 5 & -3 & 0 \\ -2 & 1 & 1 & 6 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -5 & 1 & -1 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix}.$$