Determinants of elementary matrices

- If E is obtained by interchanging two rows, then $\det(E) = -1$
- If E is obtained by multiplication of a row by k, then $\det(E) = k$
- If E is obtained by adding a multiple of one row to another row, then $\det(E) = 1$

Examples

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Let $E \cdot A$ be a matrix obtained from a square matrix A by the same elementary row operation as in E.

- If E is obtained by interchanging two rows, then $\det(E \cdot A) = -\det(A)$
- If E is obtained by multiplication of a row by k, then $\det(E \cdot A) = k \det(A)$
- If E is obtained by adding a multiple of one row to another row, then $\det(E \cdot A) = \det(A)$

Product Theorem: Let A and B be square matrices of the same size. Then

$$\det(AB) = \det(A) \cdot \det(B).$$

Example

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \qquad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$$

Example:

Let A and B be square invertible matrices. Then $B^{-1}AB$ is called the self-similar transformation of A:

$$\det(B^{-1} \cdot A \cdot B)$$

Theorem: Let A be an invertible matrix. Then, $\det(A) \neq 0$ and

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)}.$$

Example

$$A = \begin{bmatrix} -3 & a & 0 \\ 2 & 3 & -1 \\ a & 1 & -1 \end{bmatrix}$$

Theorem: Let $\det(A) \neq 0$. Then A is invertible.

Theorem: Let A be a square matrix. Then

$$\det\left(A^T\right) = \det(A)$$

<u>Example</u>

An invertible square matrix A is called **orthogonal** if $A^{-1} = A^T$. Show that $\det(A) = 1$ or $\det(A) = -1$.

<u>Example</u>

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad \theta \in \mathbb{R}$$