Diagonalization

If an $n \times n$ matrix D is **diagonal** (that is if all its entries off the main diagonal are zero), then all diagonal entries are eigenvalues:

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Powers of diagonal matrices are defined by powers of diagonal entries.

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Theorem: Let A be an $n \times n$ matrix with eigenvectors $X_1, \ldots, X_n$. Assume that the matrix $P = [X_1 \ \cdots \ X_n]$ is invertible. Then, A is diagonalizable in the sense that

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

where $\lambda, \ldots, \lambda_n$ are eigenvalues for the eigenvectors $X_1, \ldots, X_n$.

Example

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

If the matrix A is diagonalizable, the powers of A^k can be computed for any positive integer number k:

$$A = PDP^{-1}, \qquad A^k = PD^kP^{-1}.$$

Example

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Theorem: An $n \times n$ matrix A is diagonalizable if and only if every eigenvalue of multiplicity m yields m basic eigenvectors.

Example

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$