Given a point $P = P(x, y, z)$ in a three-dimensional space, the vector $\vec{OP}$ from the origin to P is called the **position vector**,

$$\vec{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Example: The position vectors of the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are unit vectors in the directions of the coordinate axes:

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Every vector $\mathbf{v}$ in the three-dimensional space is a linear combination of the basic vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Therefore, vector equality, addition, and scalar multiplications as the same operations as for column matrices.

<u>Example:</u> Given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, find the components of the vector $\vec{P_1P_2}$.

Lines

Given a point P_0 and a vector $\mathbf{d}$, there is a unique line going through the point P_0 and parallel to $\mathbf{d}$. Any other point P on this line satisfies the condition: the vector $\vec{P_0P}$ is parallel to the direction vector $\mathbf{d}$. Since "being parallel" is the same as "being a scalar multiple", the equation for the line is constructed in the parameter form:

$$\mathbf{u} = \mathbf{u}_0 + t\mathbf{d},$$

where $\mathbf{u}$ and $\mathbf{u}_0$ are the position vectors for the points P and P_0, respectively, and t is a parameter.

Example: Let A be a 3×3 matrix, B and X be 3×1 matrices. Find the rank of A so tha the general solution of $A \cdot X = B$ is a line.

Equivalent forms for lines

- Parametric vector equation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

- Parametric system of equations:

$$x = x_0 + ta,$$
$$y = y_0 + tb,$$
$$z = z_0 + tc.$$

- Symmetric form of equations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Theorem: A line is uniquely determined by two distinct points or by a single point and the direction vector.

Example: Find a parametric vector equation of the line determined by the points $P_1(1, -1, 1)$ and $P_2(2, 1, 1)$.

Example: Find points of intersections between two lines in the parameter form:

$$\begin{aligned} x &= 3 + t, \qquad & x &= 4 + 2s, \\ y &= 1 - 2t, \qquad & y &= 6 + 3s, \\ z &= 3 + 3t \qquad & z &= 1 + s \end{aligned}$$