Planes

Given a point $P_0(x_0, y_0, z_0)$ and a *nonzero* vector $\mathbf{n} = \begin{pmatrix} a & b & c \end{pmatrix}^T$, there is a unique plane through P_0, such that every vector in the plane is orthogonal to the vector $\mathbf{n}$. The vector $\mathbf{n}$ is called the **normal** vector to the plane.

Let $P(x, y, z)$ be a point that lies in the plane. Then the vector $\vec{P_0P}$ is orthogonal to the normal vector $\mathbf{n}$, such that $\mathbf{n} \cdot \vec{P_0P} = 0$. This results in the **scalar equation** for the plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Example:

$$P_0(2, 0, -1), \qquad \mathbf{n} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$$

Equivalent forms for equations of the plane

- Linear inhomogeneous equation in x,y, and z:

$$ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$.

- Vector equation for a plane

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0,$$

where

$$\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \qquad \mathbf{p}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Example:

Find an equation of the plane through the point $P_0(2, 0, -1)$ that is parallel to the plane with equation $3x + y - z = 1$.

Example:

Find the shortest distance from the point $P_1(1, -1, 3)$ to the plane with equation $x + y - 2z = 3$.