Three points define uniquely a plane in three-dimensional space if they are not located on the same line. In order to construct an equation for the plane, we need to find the normal vector to the plane from two vectors lying in the plane.

Vector product

Given two vectors $\mathbf{v}_1 = \begin{pmatrix} x_1 & y_1 & z_1 \end{pmatrix}^T$ and $\mathbf{v}_2 = \begin{pmatrix} x_2 & y_2 & z_2 \end{pmatrix}^T$, their **vector product** or **cross product** is defined as the vector

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \mathbf{k}.$$

Equivalent expression:

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{pmatrix} y_1 z_2 - y_2 z_1 \\ x_1 z_2 - x_2 z_1 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}.$$

<u>Example</u>

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Properties of the vector product:

- If $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel, $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$
- Dot product between $\mathbf{v}_3$ and $\mathbf{v}_1 \times \mathbf{v}_2$:

$$\mathbf{v}_3 \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = \det \begin{pmatrix} x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}.$$

- The vector product $\mathbf{v}_1 \times \mathbf{v}_2$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$

Example: Find an equation of the plane through the three points $P(2, -1, 0)$, $Q(0, 1, 0)$ and $R(3, 1, -2)$.

Other properties of the vector product:

- $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
- $(k\mathbf{u}) \times \mathbf{v} = k(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (k\mathbf{v})$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

Theorem: Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors with angle θ between them. Then,

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta,$$

such that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.

Example: Find the area of the triangle with vertices $P(2, -1, 0)$, $Q(0, 1, 0)$ and $R(3, 1, -2)$.