Spanning sets

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be vectors in $\mathbf{R}^n$. A vector of the form

$$\mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k,$$

where $(t_1, t_2, \ldots, t_k)$ are real parameters, is called a **linear combination** of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. The set of *all* linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called the **span** of the vectors:

$$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k : \ (t_1, t_2, \ldots, t_k)^T \in \mathbf{R}^k\} \in \mathbf{R}^n$$

Example

- $\mathbf{v} = t_1\mathbf{v}_1$
- $\mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$

Theorem

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be vectors in $\mathbf{R}^n$. The span$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a subspace of $\mathbf{R}^n$ which contains each of the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

Example

Let A be an $n \times n$ matrix with $\text{rank}(A) = r$.
Let $\{X_1, ..., X_{n-r}\}$ be basic solutions of $AX = 0$.
Let $\{C_1, ..., C_n\}$ be columns of A.

- The **null** space of A is

$$\text{null}(A) = \{X \in \mathbf{R}^n : AX = 0\} = \text{span}\{X_1, ..., X_{n-r}\}$$

- The **image** space of A is

$$\text{im}(A) = \{Y \in \mathbf{R}^n : Y = AX\} = \text{span}\{C_1, ..., C_n\},$$

Let U be a subspace of $\mathbf{R}^n$. If there exist vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ such that $U = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, then U is **spanned** by $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a **spanning set** for U.

The vector $\mathbf{u} \in \mathbf{R}^n$ belongs to U, such that $\mathbf{u} \in U$, if there exist real numbers $t_1, ..., t_k$, such that

$$\mathbf{u} = t_1\mathbf{v}_1 + ... + t_k\mathbf{v}_k.$$

The representation for $\mathbf{u} \in U$ can be expressed as the linear system

$$V\mathbf{x} = \mathbf{u},$$

where $V = [\mathbf{v}_1\mathbf{v}_2 \ldots \mathbf{v}_k]$ is the $n \times k$-matrix and $\mathbf{x} = (t_1, \ldots, t_k)^T \in \mathbf{R}^k$. When $k < n$, the linear system is over-determined and may not have solutions (which means that $\mathbf{u} \notin U$). When the system has a solution (unique or infinitely many), it means that $\mathbf{u} \in U$.

Example

Given two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in the spanning set, represent vectors $\mathbf{u}$ and $\mathbf{w}$ over the spanning set,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 8 \\ 6 \\ -3 \end{bmatrix}$$

Example

Repeat the previous example with

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -6 \\ 2 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}$$