Let A be an $m \times n$ matrix that admits two block structures:

- Column structure

$$A = [\mathbf{C}_1, ..., \mathbf{C}_n], \qquad \mathbf{C}_j \in \mathbf{R}^m$$

- Row structure

$$A = \left[\mathbf{R}_1^T, ..., \mathbf{R}_m^T\right], \qquad \mathbf{R}_j \in \mathbf{R}^n$$

such that

- Column space of A in $\mathbf{R}^m$:

$$\mathrm{col}(A) = \mathrm{span}\{\mathbf{C}_1, ..., \mathbf{C}_n\} \subset \mathbf{R}^m$$

- Row space of A in $\mathbf{R}^n$:

$$\mathrm{row}(A) = \mathrm{span}\{\mathbf{R}_1, ..., \mathbf{R}_m\} \subset \mathbf{R}^n$$

Rank Theorem

Let A be an $m \times n$ matrix and R be a row-echelon form for A. Let $r = \text{rank}(A)$ be the number of nonzero rows in R. Then:

1. The r nonzero rows of R are a basis of $\text{row}(A)$.
2. The r columns of R corresponding to the leading 1's are a basis of $\text{col}(A)$
3. $r = \dim(\text{row}A) = \dim(\text{col}A) = \text{rank}(A)$.

Example

$$A = \begin{pmatrix} 2 & 4 & -1 & 0 \\ 6 & -3 & 0 & 1 \\ 14 & -2 & -1 & 2 \end{pmatrix}$$

Properties of the rank

1. $\text{rank}(A) = \text{rank}(A^T)$
2. $\text{rank}(A) \leq \min(m, n)$
3. An $n \times n$ matrix A is invertible if and only if $\text{rank}(A) = n$.

Theorem

Let A be an $m \times n$ matrix of rank r. Then

- $\dim(\text{null}(A)) = n - r$
- $\dim(\text{im}(A)) = r$
- $\dim(\text{null}(A)) + \dim(\text{im}(A)) = n$

The basis for a subspace of $\mathbf{R}^n$ can be found from Gaussian elimination algorithm when the matrix of spanning vectors is converted to the row-echelon form. Because of duality between rows and columns, the transposed matrix can be used for the Gaussian elimination algorithm.

Example

$$U = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ -4 \\ -9 \end{pmatrix} \right\} \subset \mathbf{R}^4.$$