Basis:

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is called a **basis** of $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset \mathbf{R}^n$ if these vectors are linearly independent.

Theorem

Let the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in $\mathbf{R}^n$ be linearly independent. Given $\mathbf{u} \in \mathbf{R}^n$, the condition that $\mathbf{u} \in \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is equivalent to the condition that the set $\{\mathbf{u}, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly dependent.

Example

$$U = \text{span}\left\{ \begin{bmatrix} 2 \\ 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 14 \\ -2 \\ -1 \\ 2 \end{bmatrix} \right\}$$

Theorem

Let $k < n$ and the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in $\mathbf{R}^n$ be linearly independent. There exists a basis in $\mathbf{R}^n$ that contains the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

Example

Find the basis of $\mathbf{R}^4$ that contains the basis of U:

$$U = \text{span}\left\{ \begin{bmatrix} 2 \\ 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 14 \\ -2 \\ -1 \\ 2 \end{bmatrix} \right\}$$

Theorem

Let $U = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_l\} \subset \mathbf{R}^n$. Suppose that there exists a linearly independent set $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset U$. Then $k \leq l$.

Corollary

Let U be a subspace of $\mathbf{R}^n$. If there exist two bases in U, such that $U = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_l\} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, then $k = l$. In other words, the dimension of U is invariant with respect to bases.

Example

A standard (orthogonal and normalized) basis in $\mathbf{R}^4$:
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$.