Transformations

A **transformation** T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule such that for every vector $\mathbf{u} \in \mathbf{R}^n$ there exists a vector $T(\mathbf{u}) \in \mathbf{R}^m$. The transformation is denoted as $T : \mathbf{R}^n \mapsto \mathbf{R}^m$, where $\mathbf{R}^n$ is the **domain** of T and $\mathbf{R}^m$ is the **range** (image) of T.

Examples

- Projection from $\mathbf{R}^3$ to the (x, y)-plane

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad T(\mathbf{u}) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

- Reflection of $\mathbf{R}^3$ about the origin

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad T(\mathbf{u}) = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}.$$

A transformation $T : \mathbf{R}^n \mapsto \mathbf{R}^m$ is called **linear** if

(T1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$;

(T2) $T(k\mathbf{u}) = kT(\mathbf{u})$ for all $\mathbf{u} \in \mathbf{R}^n$ and all $k \in \mathbf{R}$.

Theorem:

Every linear transformation $T : \mathbf{R}^n \mapsto \mathbf{R}^m$ can be expressed by a matrix multiplication with an $n \times m$ matrix.

List of particular transformations

1. Projection of a vector on $\mathbf{R}^2$ to the vector $\mathbf{d} \in \mathbf{R}^2$

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \qquad P_{\mathbf{d}}(\mathbf{u}) = \mathbf{proj}_{\mathbf{d}}(\mathbf{u}) = \frac{\mathbf{d} \cdot \mathbf{u}}{\|\mathbf{d}\|^2}\mathbf{d}$$

2. Reflection of a vector on $\mathbf{R}^2$ about the vector $\mathbf{d} \in \mathbf{R}^2$

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \qquad Q_{\mathbf{d}}(\mathbf{u}) = 2P_{\mathbf{d}}(\mathbf{u}) - \mathbf{u}.$$

3. Dilation and x,y-dilations of a vector on $\mathbf{R}^2$

$$D_k(\mathbf{u}) = k\mathbf{u}$$

and

$$D_{x,k}\left(\mathbf{u}\right) = \begin{pmatrix} kx \\ y \end{pmatrix}, \qquad D_{y,k}\left(\mathbf{u}\right) = \begin{pmatrix} x \\ ky \end{pmatrix},$$

4. Translation of a vector on $\mathbf{R}^2$ in the direction of the vector $\mathbf{d} \in \mathbf{R}^2$

$$T_{\mathbf{d}}(\mathbf{u}) = \mathbf{u} + \mathbf{d}$$