Engineering Mathematics II (2M03) Tutorial I

Marina Chugunova

Department of Math. & Stat., office: HH403
e-mail: chugunom@math.mcmaster.ca

office hours: to be assigned

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Problem (1.1: 16)
Verify that the function $y(x) = 5\tan 5x$ is an explicit solution of the differential equation $y' = 25 + y^2$. Give domain of the function $y(x)$. Give at least one interval I of definition. (domain of the solution $y(x)$)

Solution

LHS: $y' = (5\tan 5x)' = \frac{25}{\cos^2 5x}$
RHS: $25 + y^2 = 25 + (5\tan 5x)^2 = 25(1 + \frac{\sin^2 5x}{\cos^2 5x}) = \frac{25}{\cos^2 5x}$
LHS = RHS (solution is verified)

Domain of the function $y = 5\tan 5x$ is the real line except points where $\cos 5x = 0, \quad x_n = \frac{\pi}{10} \pm \frac{\pi}{5}n$.

Interval I of the solution $y = 5\tan 5x$ can be chosen as $(-\frac{\pi}{10}, \frac{\pi}{10})$.

Problem (1.1: 23)
Verify that the family of functions $y = c_1e^{2x} + c_2xe^{2x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$. Assume an appropriate interval I of definition.

Solution:

$\frac{dy}{dx} = (c_1e^{2x} + c_2xe^{2x})' = (2c_1 + c_2)e^{2x} + 2c_2xe^{2x}$
$\frac{d^2y}{dx^2} = ((2c_1 + c_2)e^{2x} + 2c_2xe^{2x})' = (4c_1 + 4c_2)e^{2x} + 4c_2xe^{2x}$
$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (4c_1+4c_2)e^{2x} + 4c_2xe^{2x} - 4((2c_1+c_2)e^{2x} + 2c_2xe^{2x}) + 4(c_1e^{2x} + c_2xe^{2x}) = 0$
(solution is verified)

Interval I of the solution $y = c_1e^{2x} + c_2xe^{2x}$ can be chosen as $(-\infty, +\infty)$.

Problem (1.1: 28)
Find values of m **such that the function** $y = x^m$ **is a solution of the equation: (a)** $xy'' + 2y' = 0$ **(b)** $x^2y'' - 7xy' + 15y = 0$**. Explain your reasoning.**

Solution:

$y = x^m \quad y' = mx^{m-1} \quad y'' = m(m-1)x^{m-2}$
(a) $xy'' + 2y' = xm(m-1)x^{m-2} + 2mx^{m-1} = x^{m-1}(m^2 + m) = 0$
$m^2 + m = m(m+1) = 0, \quad m_1 = 0, \quad m_2 = -1$
two solutions are obtained: $y = 1$ **and** $y = x^{-1}$**.**

(b) $x^2y'' - 7xy' + 15y = x^2m(m-1)x^{m-2} - 7xmx^{m-1} + 15x^m = x^m(m^2 - 8m + 15) = 0$
$m^2 - 8m + 15 = 0, \quad m_1 = 3, \quad m_2 = 5$
two solutions are obtained: $y = x^3$ **and** $y = x^5$**.**

Problem (1.1: 30)
Determine whether the differential equation $y' = y^2 + 2y - 3$ possesses constant solutions. (Hint: for the constant solution $y = c$ the derivative $y' = 0$.)

Solution:

$0 = y^2 + 2y - 3, \quad y_1 = 1, \quad y_2 = -3$
the differential equation $y' = y^2 + 2y - 3$ possesses two constant solutions.

Problem (1.1: 39)
Given that $y = \sin(x)$ **is an explicit solution of the first order differential equation** $\frac{dy}{dx} = \sqrt{1 - y^2}$**. Find an interval** I **of definition. (Hint:** I **is not the interval** $-\infty < x < \infty$**)**

Solution:

LHS: $\frac{dy}{dx} = (\sin(x))' = \cos(x)$
RHS: $\sqrt{1 - y^2} = \sqrt{\cos^2 x} = |\cos(x)|$
LHS = RHS only if $\cos(x) \geq 0$**.**

The interval I **of the solution can be chosen as** $[\pi/2, \pi/2]$**.**

Problem (1.2: 8)
The second-order DE $x'' + x = 0$ possesses a two-parameter family of solutions $x = c_1 \cos t + c_2 \sin t$. Find a solution of the second-order IVP for the initial conditions: $x(\pi/2) = 0$, $x'(\pi/2) = 1$.

Solution:
Find constants c_1 and c_2 from the initial conditions:
$x(\pi/2) = c_1 \cos \pi/2 + c_2 \sin \pi/2 = c_2 = 0$
$x = c_1 \cos t$
$x' = (c_1 \cos t)' = -c_1 \sin t$
$x'(\pi/2) = -c_1 \sin \pi/2 = 1, \quad c_1 = -1$

solution of the second-order IVP is $x = -\cos t$.

Problem (1.2: 12)
The second-order DE $y'' - y = 0$ possesses a two-parameter family of solutions $y = c_1 e^x + c_2 e^{-x}$. Find a solution of the second-order IVP for the initial conditions: $y(1) = 0$, $y'(1) = e$.

Solution:

$$y = c_1 e^x + c_2 e^{-x}, \quad y' = c_1 e^x - c_2 e^{-x}$$
$$y(1) = c_1 e + c_2 e^{-1} = 0, \quad y'(1) = c_1 e - c_2 e^{-1} = e$$
$$c_1 = \frac{1}{2}, \quad c_2 = -\frac{1}{2} e^2$$

solution of the second-order IVP is $y = \frac{1}{2}(e^x - e^{2-x})$.

Problem (1.2: 18)
Determine the region of the xy-plane for which the differential equation $\frac{dy}{dx} = \sqrt{xy}$ would have a unique solution whose graph passes through a point (x_0, y_0) in the region.

Solution:

Domain of the function $\sqrt{xy}$ consists of two parts: $x \geq 0, \quad y \geq 0$ and $x \leq 0, \quad y \leq 0$.

Derivative $\frac{d}{dy}(\sqrt{xy}) = \frac{x}{2\sqrt{xy}} = \frac{1}{2}\sqrt{\left(\frac{x}{y}\right)}$. $y = 0$ is the discontinuity point.

The region for which the differential equation $\frac{dy}{dx} = \sqrt{xy}$ would have a unique solution can be taken as $x \geq 0, \quad y > 0$ or as $x \leq 0, \quad y < 0$.

Problem (1.2: 22)
Determine the region of the xy-plane for which the differential equation $(1+y^3)y' = x^2$ would have a unique solution whose graph passes through a point (x_0, y_0) in the region.

Solution:
$y' = \frac{dy}{dx} = \frac{x^2}{1+y^3}$
Domain of the function $\frac{x^2}{1+y^3}$ is $[x, y] \in [(-\infty, +\infty), (-\infty, +\infty)]$

Derivative $\frac{d}{dy}(\frac{x^2}{1+y^3}) = \frac{-3x^2y^2}{(1+y^3)^2}$

The region for which the differential equation $(1+y^3)y' = x^2$ would have a unique solution is $[x, y] \in [(-\infty, +\infty), (-\infty, +\infty)]$

Problem (1.2: 26)
Determine whether Theorem 1.1 guarantees that the differential equation $y' = \sqrt{y^2 - 9}$ possesses a unique solution through the point $(5, 3)$.

Solution:
Derivative $\frac{d}{dy}(\sqrt{y^2 - 9}) = \frac{y}{\sqrt{y^2-9}}$ has discontinuity at the point $y = 3$ and it violates the condition for the Theorem 1.1. The answer is negative.

Problem (1.2: 42)
Determine a plausible value of x_0 for which the graph of the solution of the IVP $y' + 2y = 3x - 6$, $y(x_0) = 0$ is tangent to the x-axis at $(x_0, 0)$. Explain your reasoning.

Solution:

$$y' + 2y = 3x - 6, \quad y' = 3x - 6 - 2y, \quad y'(x_0, 0) = 3x_0 - 6 = 0, \quad x_0 = 2$$

Problems (1.2: 33-34)

(33a) Verify that $3x^2 - y^2 = c$ is a one-parameter family of solutions of the differential equation $y\frac{dy}{dx} = 3x$.
(33b) Sketch the graph of the implicit solution $3x^2 - y^2 = 3$. Find all explicit solutions and give intervals I of definition for them.
(33c) The point $(-2, 3)$ is on the graph of $3x^2 - y^2 = 3$. Which explicit solution from (33b) satisfies $y(-2) = 3$.
(34a) Solve IVP $y\frac{dy}{dx} = 3x$, $y(2) = -4$ and sketch the graph of the solution.
(34b) Are there any explicit solutions of $y\frac{dy}{dx} = 3x$ that pass through the origin ?

Solution:

(33a)
(See the graph in the solution manual). Differentiating $3x^2 - y^2 = c$ with respect to x we obtain : $6x - 2y\frac{dy}{dx} = 0$. It follows from here that: $y\frac{dy}{dx} = 3x$.

(33b)

Solving $3x^2 - y^2 = 3$ for y we get:
$y_1(x) = \sqrt{3(x^2-1)}, \quad 1 < x < \infty$, $y_2(x) = -\sqrt{3(x^2-1)}, \quad 1 < x < \infty$, $y_3(x) = \sqrt{3(x^2-1)}, \quad -\infty < x < -1$, $y_4(x) = -\sqrt{3(x^2-1)}, \quad -\infty < x < -1$,

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(33c)
The answer is $y_3(x) = \sqrt{3(x^2-1)}, \quad -\infty < x < -1.$

(34a)

Find c in $3x^2 - y^2 = c$ using $y(2) = -4$.
$3 * 4 - (-4)^2 = c, \quad c = -4$**.**
The solution of IVP is $3x^2 - y^2 = -4$. To sketch the graph see (33a) in the solution manual.

(34b)

$y\frac{dy}{dx} = 3x, \quad \frac{dy}{dx} = \frac{3x}{y}$, $y = 0$ **is the point of the discontinuity of the derivative.**
The answer is negative.

See you next week :-) !