1. Finite-dimensional vector spaces

1.1. Construction of linear vector spaces and linear operators

1.1.1. Properties of linear vector space $\mathbf{V} = \mathbb{R}^n$

1. *addition*

$$\forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^n : \qquad \mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)^T$$

2. *scalar multiplication*

$$\forall \mathbf{a} \in \mathbb{R}^n, \ \lambda \in \mathbb{R} : \qquad \lambda \mathbf{a} = (\lambda a_1, \lambda a_2, ..., \lambda a_n)^T$$

3. *null-vector*

$$\mathbf{0} = (0, 0, ..., 0)^T \in \mathbb{R}^n$$

4. *norm*

$$\forall \mathbf{a} \in \mathbb{R}^n : \qquad \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}$$

5. *inner product*

$$\forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^n : \qquad (\mathbf{a}, \mathbf{b}) = a_1 b_1 + a_2 b_2 + ... + a_n b_n$$

   (a) $(\mathbf{a}, \mathbf{a}) = \|\mathbf{a}\|^2 \geq 0$

   (b) $(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{a})$

   (c) $(\mathbf{a}, \lambda \mathbf{b} + \mu \mathbf{c}) = \lambda (\mathbf{a}, \mathbf{b}) + \mu (\mathbf{a}, \mathbf{c})$

6. *dimension*

   There exists a basis of vectors $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$, such that

$$c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + ... + c_n \mathbf{u}_n = \mathbf{0} \in \mathbb{R}^n, \qquad c_1 = c_2 = ... = c_n = 0$$

7. *ortho-normal (orthogonal and normalized) basis* $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{i,j} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

1.1.2. Recipe #1: Gram-Schmidt orthogonalization procedure

The Gram Schmidt orthogonalization procedure transforms any basis $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$ in $\mathbb{R}^n$ to an ortho-normal basis $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ in $\mathbb{R}^n$.

1. Start with normalization

$$\mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$$

2. Repeat for $j = 1, 2, ..., n-1$:

   (a) orthogonalization

$$\mathbf{v}_{j+1} = \mathbf{u}_{j+1} - \alpha_1 \mathbf{e}_1 - \alpha_2 \mathbf{e}_2 - ... - \alpha_j \mathbf{e}_j,$$

   where $\alpha_i = (\mathbf{e}_i, \mathbf{v}_{j+1})$, $i = 1, 2, ..., j$

   (b) normalization

$$\mathbf{e}_{j+1} = \frac{\mathbf{v}_{j+1}}{\|\mathbf{v}_{j+1}\|}$$

Example:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{u}_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

1.1.3. Properties of projections in ortho-normal basis

$$\forall \mathbf{x} \in \mathbb{R}^n : \quad \mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + ... + x_n \mathbf{e}_n,$$

where

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{i,j}, \qquad x_j = (\mathbf{e}_j, \mathbf{x})$$

1. Invariance of inner products

$$(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 + ... + x_n y_n$$

2. Parseval's equality

$$\|\mathbf{x}\|^2 = (\mathbf{x}, \mathbf{x}) = x_1^2 + x_2^2 + ... + x_n^2$$

3. Bessel's inequality

$$\forall m \leq n : \quad x_1^2 + x_2^2 + ... + x_m^2 \leq (\mathbf{x}, \mathbf{x})$$

4. Schwarz's inequality

$$(\mathbf{x}, \mathbf{y})^2 \leq (\mathbf{x}, \mathbf{x}) \; (\mathbf{y}, \mathbf{y})$$

5. Triangle inequality

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

Theorem: Let $\mathbf{V} = \mathbb{R}^n$ be vector space with an ortho-normal basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$. Any linear operator $\mathcal{A}(v)$ from $v \in \mathbf{V}$ to $\mathcal{A}(v) \in \mathbf{V}$ is equivalent to a matrix A with elements

$$A_{i,j} = (\mathbf{e}_i, \mathcal{A}(\mathbf{e}_j))$$

such that projections of vectors $\mathbf{x} \in \mathbf{V}$ and $\mathbf{y} = \mathcal{A}(\mathbf{x}) \in \mathbf{V}$ are related by matrix multiplication:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$