1.2. Eigenvalues and eigenvectors

1.2.1. List of special matrices

1. Diagonal matrix:

$$A = D, \qquad a_{i,j} = 0, \qquad \forall i \neq j$$

2. Symmetric matrix:

$$A^T = A, \qquad a_{i,j} = a_{j,i}, \qquad \forall (i,j)$$

3. Anti-symmetric matrix:

$$A^T = -A, \qquad a_{i,j} = -a_{j,i}, \qquad \forall (i,j)$$

4. Orthogonal matrix:

$$A^T = A^{-1}, \qquad A^T A = A A^T = I$$

1.2.2. Characteristics of square matrices

1. Trace

$$\mathrm{tr}(A) = a_{11} + a_{22} + ... + a_{nn}$$

2. Determinant

$$n = 2: \quad \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

3. Rank

$$\mathrm{rank}(A)$$

which is the maximal number of linearly independent columns (or rows)

4. Inverse

$$A^{-1}: \quad A^{-1}A = AA^{-1} = I$$

1.2.3. List of important theorems

Theorem: Consider the homogeneous linear algebraic system:

$$A\mathbf{x} = \mathbf{0}, \qquad \mathbf{x} \in \mathbb{R}^n, \;\; A : \mathbb{R}^n \mapsto \mathbb{R}^n$$

There exists always zero solution: $\mathbf{x} = \mathbf{0} \in \mathbb{R}^n$. The zero solution is unique when one of the three equivalent conditions are satisfies:

- $\det(A) \neq 0$
- $\mathrm{rank}(A) = n$
- A^{-1} exists

When $\det(A) = 0$ or $\mathrm{rank}(A) < n$ or A^{-1} does not exist, there are infinitely many *non-zero* solutions of the homogeneous system.

Theorem: Consider the linear eigenvalue problem:

$$A\mathbf{x} = \lambda\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n, \;\; A : \mathbb{R}^n \mapsto \mathbb{R}^n$$

where $\lambda \in \mathbb{C}$ is the eigenvalue and $\mathbf{x} \neq \mathbf{0}$ is the eigenvector. There exists exactly n eigenvalues of A (simple or multiple, real or complex), which are given by roots of the determinant equation:

$$D(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0,$$

such that

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + ... + \lambda_n$$

and

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

Theorem: Let $A = A^T$ be a symmetric matrix in $\mathbb{R}^n$. Then,

- All eigenvalues of A are real
- There exists n linearly independent eigenvectors in $\mathbb{R}^n$
- Eigenvectors for distinct eigenvalues are orthogonal and those for multiple eigenvalues can be orthogonalized with the Gram-Schmidt orthogonalization procedure
- Eigenvectors of A form an orthonormal basis in $\mathbb{R}^n$

Theorem: Let A be an orthogonal matrix in $\mathbb{R}^n$, such that $A^T = A^{-1}$. Then,

- All eigenvalues of A have modulus one: $|\lambda_j| = 1, \forall j$
- Columns of A form an orthonormal basis in $\mathbb{R}^n$
- $\det(A) = \pm 1$

Theorem: Let A be a general square matrix in $\mathbb{R}^n$. If eigenvalues of A are distinct, then eigenvectors of A are linearly independent in $\mathbb{R}^n$.