1.3. Diagonalization and quadratic forms

1.3.1. Recipe # 2: Diagonalization of linear inhomogeneous systems

Let

$$A\mathbf{x} = \mathbf{y}, \qquad \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{b} \in \mathbb{R}^n$$

be a linear inhomogeneous system, where A is a square n-by-n matrix with n linearly independent eigenvectors $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$.

1. There exists a transformation matrix $S = [\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n]$, such that $\det(S) \neq 0$.
2. There exists a similarity transformation:

$$\forall \mathbf{x} \in \mathbb{R}^n : \quad \mathbf{x} = y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + ... + y_n\mathbf{u}_n = S\mathbf{y},$$

   such that $S^{-1}AS = D$, where D is a diagonal matrix of eigen-vlaues:

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

3. Any linear system $\mathbf{Ax} = \mathbf{b}$ becomes diagonal

$$D\mathbf{y} = \tilde{\mathbf{b}}, \qquad \tilde{\mathbf{b}} = S^{-1}\mathbf{b},$$

   with the unique solution:

$$y_j = \frac{\tilde{b}_j}{\lambda_j}, \quad j = 1, 2, ..., n.$$

Examples:

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$$

1.3.2. Properties of similarity transformations

1. $\det(S^{-1}AS) = \det(A) = \lambda_1\lambda_2 \cdot \lambda_n$
2. $\mathrm{tr}(S^{-1}AS) = \mathrm{tr}(A) = \lambda_1 + \lambda_2 + ... + \lambda_n$
3. $S^{-1}AS$ has the same eigenvalues as A

1.3.3. Recipe # 3: Diagonalization of quadratic forms

Let

$$Q(\mathbf{x}) = (\mathbf{x}, A\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}x_i x_j$$

be a quadratic form in $\mathbb{R}^n$, where A is a square symmetric n-by-n matrix, such that $A^T = A$.

1. There exists n ortho-normal eigenvectors $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$, such that $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{i,j}$.
2. There exists an orthogonal transformation matrix $S = [\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n]$, such that $\det(S) \neq 0$ and $S^T = S^{-1}$.
3. There exists an orthogonal similarity transformation:

   $$\forall \mathbf{x} \in \mathbb{R}^n : \quad \mathbf{x} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + ... + y_n\mathbf{e}_n = S\mathbf{y},$$

   such that $S^T AS = D$, where D is the matrix of eigenvalues:

   $$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

4. The quadratic form $Q(\mathbf{x})$ is diagonalized to the sum of squares:

   $$Q(\mathbf{x}) = Q(\mathbf{y}) = (\mathbf{y}, D\mathbf{y}) = \sum_{j=1}^{n} \lambda_j y_j^2.$$

Example: $4x_1^2 + 4x_1x_2 + x_2^2 = 1$