2.2. General first-order and second-order differential equations

2.2.1. Important theorem on a system of ODEs

Consider the system of linear homogeneous equations with variable coefficients:

$$\frac{d\mathbf{y}}{dx} = A(x)\mathbf{y}, \qquad t = x,$$

where $\mathbf{y} \in \mathbb{R}^n$, $A : \mathbb{R}^n \mapsto \mathbb{R}^n$, and $\mathbf{y}(0) = \mathbf{y}_0$.

Theorem: Let $\{\mathbf{y}_1(x), \mathbf{y}_2(x), ..., \mathbf{y}_n(x)\}$ be a set of n linearly independent solutions of the system for $x \geq 0$. Then,

1. The matrix of fundamental solutions

   $$S(x) = [\mathbf{y}_1(x), \mathbf{y}_2(x), ..., \mathbf{y}_n(x)]$$

   is non-singular for $x \geq 0$

2. The Wronskian determinant $W(x) = \det(S(x))$ satisfies the first-order equation:

   $$\frac{dW(x)}{dx} = \mathrm{tr}(A)(x)W(x)$$

3. The general solution of the system is

   $$\mathbf{y}(x) = c_1\mathbf{y}_1(x) + c_2\mathbf{y}_2(x) + ... + c_n\mathbf{y}_n(x),$$

   where $(c_1, c_2, ..., c_n)$ are constants in x

4. The vector of constants $\mathbf{c} = (c_1, c_2, ..., c_n)^T$ is uniquely defined from $\mathbf{y}(0) = \mathbf{y}_0$ as $\mathbf{c} = S^{-1}(0)\mathbf{y}_0$.

Example:

$$A(x) = \begin{pmatrix} 0 & 1 \\ 4/x^2 & -1/x \end{pmatrix}$$

2.2.2. Important theorem on a scalar ODE

Consider a scalar linear homogeneous n-th order equation with variable coefficients:

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = 0,$$

where $y = y(x)$ such that

$$y(0) = y_0, \quad y'(0) = y_0', \quad ... \quad y^{(n-1)}(0) = y_0^{(n-1)}.$$

Theorem: Let $\{y_1(x), y_2(x), ..., y_n(x)\}$ be a set of n linearly independent solutions of the scalar equation for $x \geq 0$. Then,

1. The Wronskian matrix is non-singular for $x \geq 0$:

$$S(x) = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

2. The Wronskian determinant $W(x) = \det(S(x))$ satisfies the first-order equation:

$$\frac{dW(x)}{dx} = -a_{n-1}(x)W(x)$$

3. The general solution of the scalar equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x),$$

   where $(c_1, c_2, ..., c_n)$ are constants in x

4. The vector of constants $\mathbf{c} = (c_1, c_2, ..., c_n)^T$ is uniquely defined from the vector of initial values $\mathbf{y}(0) = (y_0, y_0', ..., y_0^{(n-1)})^T$ as $\mathbf{c} = S^{-1}(0)\mathbf{y}_0$.

Example:

$$x^2 y'' + xy' - 4y = 0$$
$$y'' + 3y' + 2y = 0$$

2.2.3. Recipe # 7: Solution of a scalar n-order ODE with constant coefficients

Consider a scalar linear homogeneous n-th order equation with constant coefficients:

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0 y = 0,$$

where $y = y(x)$ such that

$$y(0) = y_0, \quad y'(0) = y_0', \quad ... \quad y^{(n-1)}(0) = y_0^{(n-1)}.$$

1. Look for particular solutions by separating the variables:

$$y(x) = e^{\lambda x}: \qquad D(\lambda) = 0$$

2. Find all roots of the characteristic equation $D(\lambda) = 0$:

$$\lambda = \lambda_1, \quad \lambda = \lambda_2, \quad ... \quad \lambda = \lambda_n$$

3. If all roots are distinct, construct a general solution by the Linear Superposition Principle:

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + ... + c_n e^{\lambda_n x}$$

4. Find the unique solution from the initial values:

$$\begin{aligned} c_1 + c_2 + ... + c_n &= y_0 \\ \lambda_1 c_1 + \lambda_2 c_2 + ... + \lambda_n c_n &= y_0' \\ \cdots \quad &\quad \cdots \\ \lambda_1^{n-1} c_1 + \lambda_2^{n-1} c_2 + ... + \lambda_n^{n-1} c_n &= y_0^{(n-1)} \end{aligned}$$

   or in the vector form: $\mathbf{c} = V^{-1}\mathbf{y}_0$, where V is the Vandermonde determinant of $(\lambda_1, \lambda_2, ..., \lambda_n)$.

Example:

$$\begin{aligned} y'' - p^2 y &= 0 \\ y'' + \omega_0^2 y &= 0 \\ y'' &= 0 \end{aligned}$$

2.2.4. General solution of an inhomogeneous first-order ODE

Consider the linear inhomogeneous first-order equation:

$$\frac{dy}{dx} + p(x)y = q(x), \qquad y(0) = y_0,$$

where $p(x)$ and $q(x)$ are given smooth functions of $x \geq 0$

1. When $q(x) = 0$, the solution is found by method of separation of variables:

$$y(x) = y_0 \exp\left(-\int_0^x p(s)ds\right).$$

2. When $q(x) \neq 0$, the solution is found by method of variation of constants:

$$y(x) = y_0 \exp\left(-\int_0^x p(s)ds\right)$$
$$+\left(\int_0^x q(s)\exp\left(\int_0^s p(s')ds'\right)ds\right)\exp\left(-\int_0^x p(s)ds\right).$$

3. General solution of the inhomogeneous problem is

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x)$ is a general solution of the homogeneous problem and $y_p(x)$ is a particular solution of the inhomogeneous problem

Example:

$$xy' + (1+x)y = e^{-x}, \qquad y(1) = 3.$$

Note that $x = 0$ is a singular point of the ODE.

2.2.5. General solution of an inhomogeneous second-order ODE

Consider the linear inhomogeneous second-order equation:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x), \qquad y(0) = y_0, \;\; y'(0) = y_0',$$

where $p(x)$, $q(x)$ and $r(x)$ are given smooth functions of $x \geq 0$

1. When $r(x) = 0$, the general solution is uniquely defined by two fundamental solutions $y_1(x)$ and $y_2(x)$:

   $$y(x) = c_1 y_1(x) + c_2 y_2(x),$$

   such that

   $$W(x) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \neq 0, \quad x \geq 0$$

2. When $r(x) = 0$ and one fundamental solution $y_1(x)$ is found, then the other other solution $y_2(x)$ is found from:

   $$y_2(x) = y_1(x) \int_0^x \frac{W(s)ds}{y_1^2(s)}$$

   where

   $$W(x) = \exp\left(-\int_0^x p(s)ds\right)$$

3. When $r(x) \neq 0$, the particular solution of the inhomogeneous problem:

   $$y_p(x) = \int_0^x \frac{r(s)\left(y_1(s)y_2(x) - y_1(x)y_2(s)\right)ds}{W(s)}$$

   and the general solution is

   $$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x),$$

   where c_1 and c_2 are constants.

Example:

$$x^2 y'' - 2y = x, \qquad y(1) = 1, y'(1) = -1$$

Note that $x = 0$ is a singular point of the ODE.

2.2.6. Recipe # 8: Solution of a Euler ODE

Consider a scalar linear homogeneous Euler equation:

$$x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + a_1 x \frac{dy}{dx} + a_0 y = 0,$$

where $y = y(x)$

1. Look for particular solutions by separating the variables:

$$y(x) = x^\sigma : \qquad D(\sigma) = 0$$

2. Find all roots of the characteristic equation $D(\sigma) = 0$:

$$\sigma = \sigma_1, \quad \sigma = \sigma_2, \quad ... \quad \sigma = \sigma_n$$

3. If all roots are distinct, construct a general solution by the Linear Superposition Principle:

$$y(x) = c_1 x^{\sigma_1} + c_2 x^{\sigma_2} + ... + c_n x^{\sigma_n}$$

Example:

$$\begin{aligned} x^2 y'' - 2y &= 0 \\ x^2 y'' + xy' - n^2 y &= 0 \end{aligned}$$