3 Sturm–Liouville theory

3.1 Orthogonal polynomials

3.1.1 Properties of linear vector space $V = L^2([a,b])$

$$V = \left\{ f(x) \in L^2([a,b]) : \quad \int_a^b f^2(x)dx < \infty \right\}$$

$L^2([a,b])$ is the space of square-integrable real-valued scalar functions $f(x)$ on the finite interval $x \in [a,b]$.

1. *addition*

$$\forall f(x), g(x) \in L^2([a,b]) : \qquad \exists f(x) + g(x) \in L^2([a,b])$$

2. *scalar multiplication*

$$\forall f(x) \in L^2([a,b]),\ \lambda \in \mathbb{R} : \qquad \exists \lambda f(x) \in L^2([a,b])$$

3. *null-vector*

$$f(x) = 0 \qquad \forall x \in [a,b]$$

4. *norm*

$$\forall f(x) \in L^2([a,b]) : \qquad \|f(x)\| = \left(\int_a^b f^2(x)dx \right)^{1/2}$$

5. *inner product*

$$\forall f(x), g(x) \in L^2([a,b]) : \qquad (f,g) = \int_a^b f(x)g(x)dx$$

   (a) $(f,f) = \|f(x)\|^2 \geq 0$
   (b) $(f,g) = (g,f)$
   (c) $(f, \lambda g + \mu h) = \lambda (f,g) + \mu (f,h)$

6. *dimension* - infinite-dimensional

7. *ortho-normal (orthogonal and normalized) basis* $\{e_j(x)\}_{j=1}^{\infty}$:

$$(e_i(x), e_j(x)) = \delta_{i,j} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

3.1.2 List of orthogonal polynomials

1. Legendre polynomials $P_n(x)$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad -1 \leq x \leq 1$$

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0, \quad n \neq m$$

2. Chebyshev polynomials $T_n(x)$

$$(1-x^2)y'' - xy' + n^2y = 0, \quad -1 \leq x \leq 1$$

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = 0, \quad n \neq m$$

3. Laguerre polynomials $L_n(x)$

$$xy'' + (1-x)y' + ny = 0, \quad x \geq 0$$

$$\int_{0}^{\infty} e^{-x}L_n(x)L_m(x)dx = 0, \quad n \neq m$$

4. Hermite polynomials $H_n(x)$

$$y'' - 2xy' + 2ny = 0, \quad x \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)dx = 0, \quad n \neq m$$

3.1.3 Common properties of orthogonal polynomials

- Occur in second-order scalar ODEs with regular singular points
- Represent the only solutions of the ODE which remain continuous at the regular singular points
- Can be enumerated according to the polynomial degree
- Represent an orthogonal and normalized basis in $L^2([a,b])$, subject to weighted orthogonality relations
- Satisfy the general recurrence relation

$$\phi_{n+1} + (a_n x + b_n)\phi_n + c_n \phi_{n-1} = 0, \qquad n \geq 0$$

  starting with $\phi_{-1}(x) = 0$ and $\phi_0(x) = 1$, where coefficients (a_n, b_n, c_n) are found from orthogonality relations
- Can be generated algorithmically from a generating function

Example: Legendre polynomials $P_n(x)$

- First members of the family:

$$P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{1}{2}(3x^2 - 1), \quad P_3 = \frac{1}{2}x(5x^2 - 3)$$

- Normalization relations:

$$P_n(1) = 1, \qquad \int_{-1}^{1} P_n^2(x)dx = \frac{2}{1+2n}, \quad n \geq 0$$

- Recurrence relation:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \quad n \geq 0$$

- Generating function:

$$\frac{1}{(1 - 2xh + h^2)^{1/2}} = \sum_{n=0}^{\infty} h^n P_n(x)$$