3.2 Sturm–Liouville eigenvalue problem

3.2.1 Formalism of the Sturm–Liouville theory

The Sturm–Liouville eigenvalue problem:

$$\frac{d}{dx}\left(\rho(x)\frac{dy}{dx}\right) + u(x)y + \lambda w(x)y = 0, \quad a \le x \le b,$$

subject to the boundary conditions at $x = a$ and $x = b$.

Parameters that are given:
functions $\rho(x)$, $u(x)$ and $w(x)$, where $w(x) \ge 0$ on $x \in [a, b]$
Parameters that are to be found:
eigenvalue λ, eigenfunction $y(x)$, where $y(x) \neq 0$ on $x \in [a, b]$

Theorem: Any scalar second-order ODE can be transformed to the form of the Sturm–Liouville eigenvalue problem.

Sturm–Liouville operator:

$$\mathcal{A} = -\frac{d}{dx}\left(\rho(x)\frac{d}{dx}\right) - u(x)$$

Vector space:

$$V = L^2([a, b]) : \qquad (f, g) = \int_a^b f(x)g(x)dx$$

Example: Bessel equation

$$x^2y'' + xy' + (\lambda x^2 - n^2)y = 0$$

3.2.2 Symmetric Sturm–Lioiville problems

Theorem: The Sturm–Liouville operator is symmetric with respect to the boundary conditions, such that

$$\forall f(x), g(x) \in L^2([a,b]): \qquad (f, \mathcal{A}g) = (\mathcal{A}f, g)$$

if and only if $f(x)$ and $g(x)$ satisfy symmetric boundary conditions:

$$\rho(x)\left(f(x)g'(x) - f'(x)g(x)\right)\Big|_{x=a}^{x=b} = 0$$

Example of symmetric boundary conditions:

1. Dirichlet
$$y(a) = y(b) = 0$$
2. Neumann
$$y'(a) = y'(b) = 0$$
3. Periodic
$$y(a) = y(b), \qquad y'(a) = y'(b)$$

Theorem: Let $\mathcal{A}$ be a symmetric Sturm–Liouville operator. Then,

- All eigenvalues λ are real
- There exists infinitely many linearly independent eigenfunctions $y(x)$ in vector space $L^2([a,b])$
- Eigenfunctions for distinct eigenvalues are orthogonal with respect to the weight $w(x)$ and those for multiple eigenvalues can be orthogonalized with the Gram-Schmidt orthogonalization procedure
- Eigenfunctions of $\mathcal{A}$ form an ortho-normal basis in $L^2([a,b])$

3.2.3 Series of eigenfunctions

Theorem: Any function $f(x) \in L^2([a,b])$ can be uniquely represented by the series of eigenfunctions $\{u_n(x)\}_{n=1}^{\infty}$ of the symmetric Sturm–Liouville eigenvalue problem:

$$\forall f(x) \in L^2([a,b]) : \quad f(x) = \sum_{n=1}^{\infty} c_n u_n(x),$$

where the projection formula are:

$$c_n = \frac{\int_a^b w(x) f(x) u_n(x) dx}{\int_a^b w(x) u_n^2(x) dx}, \quad n \geq 1.$$

The series converges in the mean-square sense, such that

$$\lim_{N\to\infty} \int_a^b w(x) \left(f(x) - \sum_{n=1}^{N} c_n u_n(x) \right)^2 dx = 0$$

Properties of the series of ortho-normalized eigenfunctions:

$$\int_a^b w(x) e_n(x) e_m(x) dx = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

- Parseval's equality:

$$\sum_{n=1}^{\infty} c_n^2 = \int_a^b w(x) f^2(x) dx$$

- Bessel's inequality:

$$\sum_{n=1}^{N} c_n^2 \leq \int_a^b w(x) f^2(x) dx$$

3.2.4 Recipe # 11: Solution of the boundary-value problem

$$y'' + \lambda y = 0, \qquad a \le x \le b$$

subject to two separated boundary conditions:

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \qquad \beta_1 y(b) + \beta_2 y'(b) = 0,$$

where λ is an eigenvalue, $y(x)$ is an eigenfunction ($y(x) \neq 0$), and $\alpha_{1,2}$ and $\beta_{1,2}$ are some constants. Assume that the boundary conditions are symmetric.

1. Solve the problem for $\lambda = \omega^2 > 0$ as

   $$y(x) = c_1 \cos\omega x + c_2 \sin\omega x$$

   and find the values of (c_1, c_2) and ω from the boundary conditions.

2. Solve the problem for $\lambda = 0$ as

   $$y(x) = c_1 + c_2 x$$

   and find the values of (c_1, c_2) from the boundary conditions.

3. Solve the problem for $\lambda = -p^2 < 0$ as

   $$y(x) = c_1 e^{px} + c_2 e^{-px}$$

   and find the values of (c_1, c_2) and p from the boundary conditions.

4. Write a complete set of eigenvalues $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ and a complete set of eigenfunctions $y(x) = \{u_n(x)\}_{n=1}^{\infty}$.

5. Series of eigenfunctions is

   $$f(x) \in L^2([a,b]) : \quad f(x) = \sum_{n=1}^{\infty} c_n u_n(x),$$

   where

   $$c_n = \frac{(f, u_n)}{(u_n, u_n)} = \frac{\int_a^b f(x) u_n(x) dx}{\int_a^b u_n^2(x) dx}$$