4 Partial Differential Equations

4.1 Wave and heat equations in one dimension

$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le l, \qquad t \ge 0$$

or

$$u_t = k u_{xx}, \qquad 0 \le x \le l, \qquad t \ge 0$$

subject to homogeneous boundary conditions at $x = 0$ and $x = l$ and some initial conditions at $t = 0$.

4.1.1 Recipe # 12: Solution with method of separation of variables

1. Look for product solutions by separating the variables:
$$u(x,t) = X(x)T(t)$$
and find two coupled differential equations for $X(x)$ and $T(t)$
2. Solve the Sturm–Liouville eigenvalue problem for $X(x)$ on $0 \le x \le l$, subject to the homogeneous boundary conditions at $x = 0$ and $x = l$. Obtain the set of all eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ and eigenfunctions $X(x) = \{u_n(x)\}_{n=1}^{\infty}$
3. Find a general solution of the differential equation for $T(t)$ for a particular value of $\lambda = \lambda_n$: $T(t) = c_n(t)$. The general solution will have constants of integrations.
4. By the Linear Superposition Principle, represent the general solution $u(x,t)$ with series of eigenfunctions:
$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) e_n(x),$$
where $(e_m(x), e_n(x)) = \delta_{n,m}$ and $c_n(t) = (u(x,t), e_n(x))$.
5. By the projection formula over the ortho-normal basis in $L^2([0,l])$, find the unique solution $u(x,t)$ from the initial conditions at $t = 0$.

4.1.2 Recipe # 13: Solution with the Fourier series method

1. Represent the initial condition at $t = 0$ with the Fourier series, which suits to the homogeneous boundary conditions, e.g.

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{\pi n x}{l}: \quad f(0) = f(l) = 0,$$

   where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{\pi n x}{l}dx$$

2. Represent the solution $u(x,t)$ with the same Fourier series with time-dependent Fourier coefficients:

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin\frac{\pi n x}{l}: \quad u(0,t) = u(l,t) = 0,$$

   such that $c_n(0) = b_n$.

3. Obtain the initial-value problem for Fourier coefficients $c_n(t)$.

4. Find a unique solution for $c_n(t)$ and $u(x,t)$.

Example: (heat equation)

$$u_t = u_{xx}, \qquad 0 \le x \le 1, \qquad t \ge 0$$

such that

$$u(0,t) = u(1,t) = 0$$

and

$$u(x,0) = f(x) = 1, \qquad 0 \le x \le 1$$

Eigenfunctions $e_n(x) = \sqrt{2}\sin(\pi n x)$, $n \ge 1$ are referred to as the normal modes (harmonics) of a string $0 \le x \le 1$.