5 Integral transforms

5.1 Fourier transform and applications

5.1.1 Theorem on the Fourier transform

Theorem: Let $f(x)$ be a continuously differentiable function on $x \in \mathbb{R}$, which is absolutely integrable, so that

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

Then, $f(x)$ can be replaced by the Fourier integral:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk,$$

where $\hat{f}(k)$, $k \in \mathbb{R}$ is the Fourier transform:

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

The Fourier transform converges uniformly for any $x \in \mathbb{R}$, $k \in \mathbb{R}$.

Examples:
Gaussian pulse:

$$f(x) = \frac{1}{a} \exp\left(-\frac{x^2}{2a^2}\right), \qquad \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{a^2k^2}{2}\right)$$

Lorentzian pulse:

$$f(x) = \exp\left(-\frac{|x|}{a}\right), \qquad \hat{f}(k) = \frac{a}{\pi(1 + a^2k^2)}$$

5.1.2 Properties of the Fourier transform

- Fourier transform is a linear operator

$$
\begin{aligned}
(i)\quad & \mathcal{F}[f(x)+g(x)] = \mathcal{F}[f(x)] + \mathcal{F}[g(x)] \\
(ii)\quad & \mathcal{F}[\lambda f(x)] = \lambda \mathcal{F}[f(x)]
\end{aligned}
$$

- shift

$$\mathcal{F}[f(x-a)] = e^{-ika}\hat{f}(k)$$

- derivatives

$$\mathcal{F}[f^{(n)}(x)] = (ik)^n \hat{f}(k)$$

- powers

$$\mathcal{F}[x^n f(x)] = i^n \hat{f}^{(n)}(k)$$

- convolution

$$\mathcal{F}[f(x) * g(x)] = \mathcal{F}\left[\int_{-\infty}^{\infty} f(s)g(x-s)ds\right] = 2\pi \hat{f}(k)\hat{g}(k)$$

- Parseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 2\pi \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk$$

- Dirac delta-function

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

such that for any continuous function $f(x)$ on $x \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

and

$$\mathcal{F}[\delta(x)] = \frac{1}{2\pi}, \qquad \mathcal{F}[1] = \delta(k)$$

5.1.3 Recipe # 14: Solutions of PDEs with the Fourier transform

$$u_t = u_{xx}, \qquad x \in \mathbb{R}, \quad t \geq 0$$

such that

$$u(x,0) = f(x), \qquad x \in \mathbb{R}$$

where $f(x)$ is continuously differentiable and absolutely integrable.

1. Represent $f(x)$ with the Fourier transform:

   $$u(x,0) = f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

   where

   $$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

2. Represent the solution $u(x,t)$ with the time-dependent Fourier transform:

   $$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(k,t) e^{ikx} dk$$

   such that $\hat{u}(k,0) = \hat{f}(k)$.

3. Apply Fourier transform to the PDE and obtain the initial-value problem for $\hat{u}(k,t)$ in t:

   $$\mathcal{F}[u_t(x,t)] = \frac{\partial \hat{u}(k,t)}{\partial t}, \qquad \mathcal{F}[u_{xx}(x,t)] = (ik)^2 \hat{u}(k,t)$$

4. Find a unique solution for $\hat{u}(k,t)$ and $u(x,t)$:

   $$u(x,t) = \int_{-\infty}^{\infty} \hat{f}(k) e^{-k^2 t} e^{ikx} dk$$

5. Apply the convolution formula and rewrite the solution $u(x,t)$:

   $$u(x,t) = \frac{1}{4\pi t} \int_{-\infty}^{\infty} f(s) \exp\left(-\frac{(x-s)^2}{4t}\right) ds$$