MATH 3C03 Home Work # 5

(Q1) $\begin{cases} \Delta u = 0 & \text{inside a sphere of radius } R \\ u|_{r=R} = 5\cos^2(\theta) \end{cases}$

Since the boundary data does not depend on the azimutal angle φ, the solution does not depend on φ.

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0.$$

Recall Legendre polynomials

$$\begin{cases} P_0(x) = 1 \\ P_1(x) = x \\ P_2(x) = \frac{1}{2}(3x^2 - 1) \end{cases} \qquad x = \cos(\theta) \qquad \begin{cases} P_0 = 1 \\ P_1 = \cos(\theta) \\ P_2 = \frac{1}{2}(3\cos^2\theta - 1) \end{cases}$$

Expand the boundary data as a linear combination of Legendre polynomials at $x = \cos(\theta)$:

$$u|_{r=R} = 5\cos^2(\theta) = \frac{5}{3}(2P_2 + P_0)$$

$$2P_2 = 3\cos^2(\theta) - 1 = 3\cos^2(\theta) - P_0 \Rightarrow \cos^2(\theta) = \frac{2P_2 + P_0}{3}$$

Now look for solution as a sum of spherical harmonics

$$u(r,\theta) = A_0(r)\, P_0(\cos\theta) + A_2(r)\, P_2(\cos\theta)$$

where $A_0(R) = 5/3 \qquad A_2(R) = 10/3$

Recall that $\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left[\sin\theta \dfrac{dP_m(\cos\theta)}{d\theta}\right] = -m(m+1)\, P_m(\cos\theta)$

Therefore

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{dA_0}{dr}\right]P_0 = -\frac{A_0}{r^2 \sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{dP_0}{d\theta}\right] = 0 \qquad m=0$$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{dA_2}{dr}\right]P_2 = -\frac{A_2}{r^2 \sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{dP_2}{d\theta}\right] = \frac{6}{r^2}A_2 P_2 \qquad m=2$$

Q1 end

$$\frac{d}{dr}\left[r^2 \frac{dA_0}{dr}\right] = 0 \Rightarrow r^2 \frac{dA_0}{dr} = C_1$$

$$\frac{dA_0}{dr} = \frac{C_1}{r^2} \Rightarrow A_0 = -\frac{C_1}{r} + C_2$$

$C_1 = 0$ to have $|A_0(0)| < \infty$

$C_2 = 5/3$ to have $A_0(R) = 5/3$

$$\frac{d}{dr}\left[r^2 \frac{dA_2}{dr}\right] = 6 A_2 \Rightarrow r^2 \frac{d^2 A_2}{dr^2} + 2r \frac{dA_2}{dr} - 6A_2 = 0$$

Euler's equation

$A_2 = r^\sigma$: $\quad \sigma(\sigma-1) + 2\sigma - 6 = 0$

$\sigma(\sigma+1) = 6 = 2(2+1) \Rightarrow \sigma_1 = 2, \quad \sigma_2 = -3$

$A_2(r) = \frac{C_1}{r^3} + C_2 r^2 \qquad C_1 = 0$ to have $|A_2(0)| < \infty$

$C_2 = \frac{10}{3R^2}$ to have $A_2(R) = 10/3$

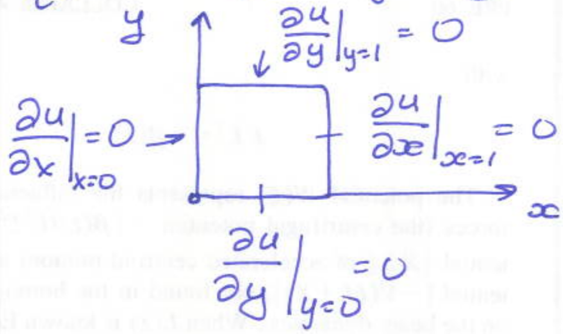
The resulting solution is

$$u(r,\theta) = \frac{5}{3} P_0 + \frac{10}{3}\frac{r^2}{R^2} P_2(\cos\theta)$$

$$= \frac{5}{3} + \frac{10}{3}\frac{r^2}{R^2}\frac{1}{2}\left(3\cos^2\theta - 1\right)$$

$$= \frac{5}{3}\left(1 - \frac{r^2}{R^2}\right) + 5\frac{r^2}{R^2}\cos^2\theta.$$

Q2 $\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial D} = 0 \\ u\big|_{t=0} = \cos(\pi x)\cos^3(\pi y) \end{cases}$ in $D = \{0 < x < 1,\ 0 < y < 1\}$



Expand the initial data over the Fourier cosine harmonics, which satisfy Neumann boundary conditions.

$$\cos^3(\pi y) = \left(\frac{e^{i\pi y} + e^{-i\pi y}}{2}\right)^3 = \frac{e^{3i\pi y} + 3e^{i\pi y} + 3e^{-i\pi y} + e^{-3i\pi y}}{8}$$

$$= \frac{1}{4}\cos(3\pi y) + \frac{3}{4}\cos(\pi y)$$

Hence,

$$u\big|_{t=0} = \frac{3}{4}\cos(\pi x)\cos(\pi y) + \frac{1}{4}\cos(\pi x)\cos(3\pi y)$$

Solution as a sum of Fourier cosine harmonics

$$u(x,y,t) = A_1(t)\cos(\pi x)\cos(\pi y) + A_2(t)\cos(\pi x)\cos(3\pi y)$$

$$A_1(0) = 3/4 \qquad A_2(0) = 1/4$$

$$\frac{dA_1}{dt} = -2\pi^2 A_1 \qquad \frac{dA_2}{dt} = -10\pi^2 A_2 \implies \begin{cases} A_1(t) = \frac{3}{4}e^{-2\pi^2 t} \\ A_2(t) = \frac{1}{4}e^{-10\pi^2 t} \end{cases}$$

The resulting solution is

$$u(x,y,t) = \frac{3}{4}e^{-2\pi^2 t}\cos(\pi x)\cos(\pi y) + \frac{1}{4}e^{-10\pi^2 t}\cos(\pi x)\cos(3\pi y).$$

(Q3) The inverse Fourier transform of

$$\hat{f}(k) = \frac{2}{8+4k+k^2} = \frac{2}{4+(k+2)^2}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2e^{ikx}}{4+(k+2)^2}\, dk$$

$$= \frac{2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i(k+2)x}\cdot e^{-2ix}}{4+(k+2)^2}\, dk = \frac{2e^{-2ix}}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{4+k^2}\, dk$$

$$k+2 = \tilde{k}$$
$$dk = d\tilde{k}$$

Recall that the Fourier transform of $e^{-|x|/a}$ is $\frac{2a}{1+a^2k^2}$.

Hence, the inverse Fourier transform of $\frac{4}{4+k^2}$ $(a = 1/2)$ is $e^{-2|x|}$

This yields

$$f(x) = \frac{1}{2}e^{-2ix}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{4+k^2}\, e^{ikx}\, dk = \frac{1}{2}e^{-2ix-2|x|}$$

(Q4) Double Fourier transform of

$$f(x,y) = \frac{1}{4\pi t}\, e^{-\frac{x^2+y^2}{4t}}, \quad t>0$$

$$\hat{f}(k,p) = \iint_{-\infty}^{\infty} f(x,y)\, e^{-ikx-ipy}\, dx\, dy$$
$$= \frac{1}{4\pi t}\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}-ikx}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}-ipy}\, dy\right)$$

Recall that the Fourier transform of $\frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}$ is e^{-tk^2}

Hence

$$\hat{f}(k,p) = \left(\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}-ikx}\, dx\right)\left(\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}-ipy}\, dy\right)$$
$$= e^{-t(k^2+p^2)}.$$

(Q5)
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R} \\ u|_{t=0} = e^{-x^2} \end{cases}$$

Using Fourier transform, if $f(x) = e^{-x^2}$, then

$$\hat{f}(k) = \sqrt{\pi}\, e^{-k^2/4}$$ (take $t = 1/4$ in the Fourier transform of $\frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$)

Then
$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t) e^{-ikx}\, dx$$

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} = -k^2 \hat{u} \\ \hat{u}|_{t=0} = \hat{f}(k) \end{cases} \Rightarrow \hat{u}(k,t) = \hat{f}(k) e^{-tk^2}$$

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k,t) e^{ikx}\, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{-tk^2 + ikx}\, dk$$

$$= \frac{1}{2\pi} \sqrt{\pi} \int_{-\infty}^{\infty} e^{-k^2(t+1/4) + ikx}\, dk$$

Denote $t + 1/4 = \tau$ and use the inverse Fourier transform of $e^{-\tau k^2}$ to be $\frac{1}{\sqrt{4\pi\tau}} e^{-\frac{x^2}{4\tau}}$

$$= \frac{\sqrt{\pi}}{\sqrt{4\pi\tau}} e^{-\frac{x^2}{4\tau}}$$

$$= \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2}{1+4t}}, \quad t \geq 0, \quad x \in \mathbb{R}.$$

(Q6) $$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \alpha\,\delta(x), & x \in \mathbb{R},\ t>0 \\ u\big|_{t=0} = 0 \end{cases}$$

Recall the Fourier transform of Dirac delta distribution:

$$\hat{\delta}(k) = 1, \quad k \in \mathbb{R}.$$

Using the Fourier transform

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-ikx}\, dx$$

we obtain the differential equation for $\hat{u}(k,t)$:

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} = -k^2 \hat{u} + \alpha, & t>0 \\ \hat{u}\big|_{t=0} = 0 \end{cases}$$

The general solution is

$$\hat{u}(k,t) = \frac{\alpha}{k^2} + c\, e^{-tk^2}$$

and c is found from the initial condition:

$$\hat{u}(k,0) = 0 = \frac{\alpha}{k^2} + c \implies c = -\frac{\alpha}{k^2}$$

Hence $$\hat{u}(k,t) = \frac{\alpha}{k^2}\left(1 - e^{-tk^2}\right)$$

In the Fourier transform form, the solution is

$$u(x,t) = \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-tk^2}}{k^2}\, e^{ikx}\, dk.$$

Note that

$$\frac{\partial u}{\partial t} = \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} e^{-tk^2} \cdot e^{ikx}\, dk = \frac{\alpha}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}, \quad t>0.$$

Hence,

$$u(x,t) = \alpha \int_0^t \frac{1}{\sqrt{4\pi s}}\, e^{-\frac{x^2}{4s}}\, ds,$$

where $\dfrac{1}{\sqrt{s}}$ is integrable near $s=0$.