1 Theory of function of complex variable

1.1 Cauchy–Riemann equations

1.1.1 Properties of linear vector space $\mathbb{C}$

$$(x, y) \in \mathbb{R}^2, \quad z \in \mathbb{C}: \qquad z = x + iy = re^{i\theta},$$

where

$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z),$$

and

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(y/x)$$

1. addition
$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
2. scalar multiplication
$$\lambda z = (\lambda x) + i(\lambda y)$$
3. Null vector
$$z = 0 + i0$$
4. multiplication
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$
5. division
$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{y_1 x_2 - y_2 x_1}{x_2^2 + y_2^2}$$
6. complex conjugation
$$\bar{z} = x - iy,$$
7. inner product and the norm
$$(z_1, z_2) = \bar{z}_1 z_2, \qquad |z|^2 = (z, z) = \bar{z} z = x^2 + y^2$$

1.1.2 Recipe # 1: complex roots of algebraic equations

Consider an algebraic equation of degree n:

$$z^n = a, \qquad a \in \mathbb{C}.$$

By Fundamental Theorem of Algebra, there exist n complex roots z_1, z_2, ..., z_n that satisfy the algebraic equation.

1. Use the polar form for an unknown root:

$$z = re^{i\theta}, \qquad r \geq 0, \quad 0 \leq \theta < 2\pi$$

2. Express the parameter a in the polar form by using periodicity of the argument with period of 2π:

$$a = |a|e^{i\alpha + 2\pi ki}, \qquad k = 0, \pm 1, \pm 2, ...$$

3. Find the unique value for r:

$$r = |a|^{1/n}$$

4. Find the most general representation for θ:

$$\theta = \frac{\alpha + 2\pi k}{n}.$$

5. The n fundamental roots are defined for $k = 0, 1, ..., n-1$. All other roots (for $k \geq n$ and $k < 0$) reduce to the fundamental roots due to periodicity of the argument.

Examples:

$$z^2 + 1 = 0$$

$$z^4 + 1 = 0$$

$$z^4 - 1 = 0$$

1.1.3 Definitions of function of complex variable

$$z = x + iy, \qquad w = f(z) = u(x, y) + iv(x, y)$$

1. *Domain* is an open connected region in the z-plane, where $f(z)$ is defined
2. *Range* is a region in the w-plane, where $f(z)$ is defined
3. The function $f(z)$ is *continuous* at the point $z = z_0$ if there exists $\lim_{z \to z_0} f(z) = f(z_0)$
4. The function $f(z)$ is *continuously differentiable* at the point $z = z_0$ if there exists the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0).$$

Examples:

- Polynomials and power functions

$$f(z) = z^4, \qquad g(z) = z^{1/4}$$

- Exponential and logarithmic functions

$$f(z) = e^{\lambda z}, \qquad g(z) = \ln(z)$$

- Rational functions

$$f(z) = \frac{z^4 - z^2 - 2}{z^2 + 1}, \qquad f(z) = \frac{1}{z^4 + 1}$$

- Non-analytic functions (depend on $\bar{z}$)

$$f(z) = \frac{\bar{z}}{z}, \qquad f(z) = z + \bar{z}$$

1.1.4 The Cauchy–Riemann Theorem

Theorem: Let $w = f(z) = u(x,y) + iv(x,y)$ be a function of $z = x+iy$ in a domain $D \subset \mathbb{C}$. The function $f(z)$ is differentiable at the point $z_0 \in D$ if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at the point $x = x_0$ and $y = y_0$, such that the derivative of $f(z)$ is

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Definition: If the function $f(z)$ is differentiable at $z \in D$, then $f(z)$ is said to be *analytic* in $z \in D$. Analytic functions $f(z)$ depend on z, but do not depend on $\bar{z}$. If the function $f(z)$ is differentiable at any $z \in \mathbb{C}$, then $f(z)$ is said to be *entire*.

Theorem: Let functions $u(x,y)$ and $v(x,y)$ satisfy the Cauchy–Riemann equations in $z \in D \subset \mathbb{C}$. These functions satisfy the Laplace equations:

$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0, \qquad (x,y) \in D$$

and are referred to as *conjugate harmonic* functions. Families of contour curves $u(x,y) = u_0$ and $v(x,y) = v_0$ are orthogonal to each other.

1.1.5 Classification of singular points of the function $f(z)$

Consider a function $f(z)$ that depends on $z = x + iy$ and does not depend on $\bar{z} = x - iy$. The point $z = z_0$ is a regular point of $f(z)$ if $f(z)$ is analytic at $z = z_0$. In the opposite case, the point $z = z_0$ is a singular point of $f(z)$.

1. Isolated singular points
   - Poles
   - Essential singularities
2. Non-isolated singular points
   - Branch points
   - Accumulation points

Examples:

Isolated singular points:

$$f(z) = \frac{z}{1+z^2}, \qquad f(z) = e^{1/z}$$

Non-isolated singular points:

$$f(z) = (z-1)^{1/3}, \qquad f(z) = \cot(1/z)$$