2.2 Location of zeros and poles in a complex plane

2.2.1 Theorems on analytic functions

Theorem (Maximum Modulus Principle)): If $f(z)$ is analytic in a bounded domain $D \subset \mathbb{C}$ and γ is the boundary of D. Then,

$$\forall z \in D : \quad |f(z)| \leq \max_{z \in \gamma} |f(z)|$$

Theorem (Liouville): If $f(z)$ is analytic everywhere on $z \in \mathbb{C}$ and in the limit $|z| \to \infty$, then $f(z)$ is constant.

Fundamental Theorem of Algebra: If $f(z)$ is a polynomial of degree N, then it has N complex zeros (roots) on $z \in \mathbb{C}$.

Theorem: If $f(z)$ is analytic in a bounded domain $D \subset \mathbb{C}$, then it may have only finitely many zeros inside D.

Theorem (Argument Principle): Let $f(z)$ be analytic in a bounded domain $D \subset \mathbb{C}$ and have no zeros on the boundary γ. Then,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)dz}{f(z)} = N,$$

where N is the number of zeros of $f(z)$ inside D.

Theorem (Rouche): Let $f(z)$ and $g(z)$ be analytic in a bounded domain $D \subset \mathbb{C}$ and $|f(z)| < |g(z)|$ on the boundary γ. Then, $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside D.

2.2.2 Recipe # 7: Prediction of zeros of analytic functions

Given analytic function $f(z)$ in a domain $D \subset \mathbb{C}$, find the number of zeros of $f(z)$ in D.

1. Consider a boundary of D as a curve γ. Compute the change of the argument of $f(z)$ along the closed curve γ, denoted as $[\arg f(z)]_\gamma$. By argument principle, the number of zeros is $[\arg f(z)]_\gamma / 2\pi$.
2. Consider $f(z) = g(z) + h(z)$, such that $|g(z)| < |h(z)|$ and the number of zeros of $h(z)$ in D is known. By the Rouche Theorem, the numbers of zeros of $f(z)$ and $h(z)$ are equal.

Examples:

$$f(z) = z^5 + 1 \quad \text{in the first quadrant}$$

$$f(z) = e^z - 4z - 1 \quad \text{in the unit disc}$$

Mittag–Leffler expansion (for meromorphic functions):

$$f(z) = \pi \cot(\pi z) = \sum_{m \in \mathbb{Z}} \frac{z}{z^2 - m^2}$$

Weierstrass factorization (for analytic funcitons):

$$f(z) = \frac{\sin \pi z}{\pi z} = \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{m^2}\right)$$