3 Theory of probability

3.1 Discrete random variables

3.1.1 Definition of probability

1. **Common sense:** Probability is the quantity that measures likeness of an event.
2. **Theory:** If an event consists of m outcomes out of n possible outcomes, probability of the event is m/n. (All outcomes are assumed to occur equally likely.)
3. **Experiment:** If an event occurs $m(n)$ times in n independent trials, probability of the event is
$$\lim_{n\to\infty} \frac{m(n)}{n}$$
(All events are assumed to occur equally likely at any time.)

3.1.2 Properties of probability

The set of n possible outcomes $\{E_1, E_2, ..., E_n\}$ is called the sample space. Probability of E_j is denoted as $p(E_j)$.

1. If E_j is impossible, then $p(E_j) = 0$
2. If E_j is the only possibility, then $p(E_j) = 1$
3. In general, $0 \leq p(E_j) \leq 1$ and
$$\sum_{j=1}^{n} p(E_j) = 1$$
4. If two events E_i and E_j are mutually exclusive, then
$$P(E_i \cap E_j) = 0, \qquad P(E_i \cup E_j) = P(E_i) + P(E_j)$$
5. If two events E_i and E_j are independent and their outcomes are considered after another, then
$$P(E_i, E_j) = P(E_i)P(E_j)$$

3.1.3 Properties of probability distributions

If X runs through a set of discrete outcomes $\{x_1, ..., x_n\}$, which are defined with some probability, X is called a discrete random variable. The function $p(x)$ of probabilities of $x = \{x_1, ..., x_n\}$ is called the probability distribution.

1. Normalization condition:

$$\sum_{j=1}^{n} p(x_j) = 1$$

2. Mean value of x (the most expected value of x):

$$\mu = \sum_{j=1}^{n} x_j p(x_j)$$

3. Moments of distribution:

$$m_n = \sum_{j=1}^{n} x_j^n p(x_j)$$

   such that $m_1 = \mu$.

4. Variance (the spread of $p(x)$ near the mean value):

$$\sigma^2 = \sum_{j=1}^{n} (x_j - \mu)^2 p(x_j) = m_2 - \mu^2,$$

   where σ is standard deviation from the mean.

Example:

$$p(n) = \frac{1}{2^n}, \qquad n \geq 1$$

3.1.4 Important distributions of discrete random variables

1. Binomial distribution

$$p(k) = \frac{n!}{k!(n-k)!} p^k q^{n-k}, \qquad 0 \le k \le n,$$

where $0 \le p \le 1$, $q = 1 - p$, and $n \in \mathbb{N}$.

(a) Normalization condition:

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k q^{n-k} = (p+q)^n = 1$$

(b) Mean value and variance

$$\mu = np, \qquad \sigma^2 = npq.$$

2. Poisson distribution

$$p(k) = \frac{\mu^k}{k!} e^{-\mu}, \qquad k \ge 0,$$

where μ is parameter.

(a) Normalization condition:

$$\sum_{k=0}^{\infty} \frac{\mu^k}{k!} e^{-\mu} = e^{\mu} e^{-\mu} = 1$$

(b) Mean value and variance

$$\mu = \mu, \qquad \sigma^2 = \mu.$$