3.2 Continuous random variables

3.2.1 Properties of probability density

$$\text{Prob}\{a \leq x \leq b\} = \int_a^b p(x)dx,$$

where $p(x)$ is probability density function (pdf).

1. Range

$$0 \leq p(x) \leq 1, \qquad x \in \mathbb{R}$$

2. Normalization

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

3. Mean value on $x \in \mathbb{R}$:

$$\mu = \int_{-\infty}^{\infty} xp(x)dx$$

4. Moments

$$m_n = \int_{-\infty}^{\infty} x^n p(x)dx,$$

   such that $m_1 = \mu$.

5. Variance

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)dx = m_2 - \mu^2,$$

   where σ is standard deviation from the mean.

Example:

$$p(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 2-x, & 1 \leq x \leq 2 \\ 0, & 2 \leq x \end{cases}$$

3.2.2 Normal (Gaussian) distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

where μ is the mean and σ is the standard deviation.
Standardized normal distribution: $\mu = 0$ and $\sigma = 1$ in the standardized variable:

$$z = \frac{x-\mu}{\sigma}$$

1. Normalization
$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz = 1$$
2. Mean value:
$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\mu+\sigma z) e^{-\frac{1}{2}z^2} dz = \mu$$
3. Variance
$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz = \sigma^2$$
4. Confidence intervals:
$$\begin{aligned} \text{Prob}\{\mu-\sigma \le x \le \mu+\sigma\} &= 68\% \\ \text{Prob}\{\mu-2\sigma \le x \le \mu+2\sigma\} &= 95.5\% \\ \text{Prob}\{\mu-3\sigma \le x \le \mu+3\sigma\} &= 99.7\% \end{aligned}$$
5. Characteristic function
$$\phi(s) = \int_{-\infty}^{\infty} p(x) e^{isx} dx = e^{is\mu} e^{-\frac{1}{2}\sigma^2 s^2}$$
such that $\phi(0) = 1$, $\phi'(0) = i\mu$, and $\phi^{(n)}(0) = i^n m_n$
6. Central moments
$$M_n = \int_{-\infty}^{\infty} (x-\mu)^n p(x) dx = \begin{cases} \frac{(-1)^{k+1}(2k)!\sigma^{2k}}{2^k k!}, & n = 2k \\ 0, \quad n = 2k+1 \end{cases}$$
such that $M_2 = \sigma^2$ and $M_4 = -3\sigma^4$.

3.2.3 Recipe # 10: How to find probability from the normal distribution

Define the standardized normal cumulative distribution:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}s^2} ds$$

Find the probability $\text{Prob}\{x \leq x_0\}$ in terms of the value of $\Phi(z)$.

1. Use the standardized variable and transform:

$$\text{Prob}\{x \leq x_0\} = \Phi\left(\frac{x_0 - \mu}{\sigma}\right).$$

2. Compute $\Phi(z_0)$ from the table of $\Phi(z)$. If necessary, use the identity:

$$\Phi(z) + \Phi(-z) = 1, \qquad \forall z \in \mathbb{R}.$$

3.2.4 χ-square distribution of degree $n \geq 1$

Probability density funciton:

$$p(u) = \frac{u^{(n-2)/2} e^{-u/2}}{2^{n/2}\Gamma(n/2)}, \qquad u > 0$$

Characteristic function:

$$\phi(s) = \int_0^{\infty} p(x) e^{-sx} dx = \frac{1}{(1+2s)^{n/2}}$$

such that $\phi(0) = 1$, $\phi'(0) = -n$, and $\phi''(0) = n(n+2)$. As a result,

$$\mu = n, \qquad \sigma^2 = 2n$$