5 Calculus of variations

5.1 Variational problems in one dimension

5.1.1 Euler-Lagrange equations

Consider a curve $y = y(x)$ on the plane (x, y) between two points (a, α) and (b, β) and a functional on the curve $y(x)$:

$$I = \int_a^b F(x, y, y')dx,$$

where $F(x, y, y')$ is a scalar function of (x, y, y').

Theorem: The curve $y(x) \in C^2([a, b])$ gives an extremal value of I if it solves the Euler-Lagrange differential equation:

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0.$$

Theorem: If $F = F(y, y')$, the Euler–Lagrange equation can be integrated as follows:

$$F(y, y') - y'\frac{\partial F}{\partial y'} = F_0 = \text{const}$$

Examples:

- Shortest curve between two points

$$(\text{min}) \;\; L = \int_a^b \sqrt{1 + y'^2}dx$$

- Surface of revolution with smallest area

$$(\text{min}) \;\; S = 2\pi \int_a^b y\sqrt{1 + y'^2}dx$$

- Fermat's principle of shortest travel time of a light ray

$$(\text{min}) \;\; T = \frac{1}{c}\int_a^b n(x, y)\sqrt{1 + y'^2}dx$$

5.1.2 Classical mechanics

Consider a system of n particles with a set of generalized coordinates $\{q_1, q_2, ..., q_n\}$ and a set of generalized velocities $\{\dot{q}_1, \dot{q}_2, ..., \dot{q}_n\}$.

Hamilton's Principle: The system of particles move from an initial time $t = 0$ to the final time $t = T$ according to the least action,

$$\text{(min)} \;\; S = \int_0^T L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n, t)dt,$$

where L is the Lagrangian density and S is the action functional. As a result, the Euler-Lagrange equations of motion are

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0, \quad j = 1, ..., n$$

Theorem: Let the Lagrangian L be independent on a particular coordinate q_i. There exists a constant of motion:

$$P_i = \frac{\partial L}{\partial \dot{q}_i} = \text{const}$$

Examples:

- A particle in a three-dimensional potential field

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z)$$

- A particle in a two-dimensional radially-symmetric field

$$\begin{aligned} L &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - V(\sqrt{x^2 + y^2}) \\ &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \end{aligned}$$

5.1.3 Hamiltonian mechanics

Theorem: Let the Lagrangian L be independent of t. The energy of the system conserves in time:

$$E = \sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n) = \text{const}$$

Let $(p_1, ..., p_n)$ be a set of generalized momenta:

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \quad j = 1, ..., n$$

Consider a transformation from a set $(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n)$ to the set $(q_1, ..., q_n, p_1, ..., p_n)$ by using the Legendre transformation:

$$H(q_1, ..., q_n, p_1, ..., p_n) = \sum_{j=1}^{n} p_j \dot{q}_j - L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n, t)$$

The Hamilton equations of motions are

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad j = 1, ..., n$$

Examples:

- A particle in a three-dimensional potential field

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(x, y, z)$$

- A particle in a two-dimensional radially-symmetric field

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r)$$

$$E = \frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2} + V(r)$$