5.2 Variational problems in many dimensions

5.2.1 PDEs as extremal values of functionals

- Laplace equation in $\mathbb{R}^2$

$$u_{xx} + u_{yy} = 0, \qquad (x, y) \in D \subset \mathbb{R}^2,$$

such that

$$u\Big|_{\partial D} = \phi(x, y)$$

Solutions of the Dirichlet problem for the Laplace equation give minimal values of the energy functional

$$(\min) \;\; W = \frac{1}{2} \int \int_{(x,y)\in D} \left(u_x^2 + u_y^2\right) dxdy$$

- Wave equation in $\mathbb{R}^1$

$$u_{tt} - c^2 u_{xx} = 0, \qquad a \le x \le b, \;\; 0 \le t \le T$$

such that

$$u(a, t) = u(b, t) = 0, \qquad 0 \le t \le T$$

and

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \qquad a \le x \le b$$

Solutions of the Dirichlet problem for the wave equation give minimal values of the action functional:

$$(\min) \;\; S = \int_0^T L[u]dt, \qquad L[u] = \frac{1}{2} \int_a^b \left(u_t^2 - c^2 u_x^2\right) dx$$

The energy functional of the wave equation is

$$H[u, p] = \frac{1}{2} \int_a^b \left(p^2 + u_x^2\right) dx, \qquad p = u_t$$

5.2.2 Eigenvalue problems

Consider the eigenvalue problem in two dimensions:

$$\phi_{xx} + \phi_{yy} + \lambda\phi = 0, \qquad (x, y) \in D \subset \mathbb{R}^2,$$

such that

$$\phi\big|_{\partial D} = 0$$

Solutions of the Dirichlet eigenvalue problem give minimal values of the Rayleigh–Ritz functional

$$\Lambda = \frac{\int\int_D \left(\phi_x^2 + \phi_y^2\right) dxdy}{\int\int_D \phi^2 dxdy}$$

There exists infinitely many eigenvalues of the Dirichlet problem

$$\lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$$

where the minimal eigenvalue λ_0 is always simple and is referred to as the ground state. The variational method is useful for approximation of the ground state. Constrained minimization and knowledge of the eigenfunctions for smaller eigenvalues is needed to find larger eigenvalues.

Example:

$$\phi'' + \lambda\phi = 0, \qquad 0 \leq x \leq \pi$$

such that

$$\phi(0) = \phi(\pi) = 0.$$

The exact solution is

$$\lambda = n^2, \quad \phi(x) = \sin(nx), \quad n \in \mathbb{N}$$

and the ground state is $\lambda_0 = 1$.