MATH 3J04: Solutions to Home Assignment # 2

Problem 3.3 #8: The spectrum of the underlying matrix consists of a single eigenvalue $\lambda_1 = -6$ and a double eigenvalue $\lambda_2 = \lambda_3 = -3$. There are three (non-degenerate) eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. The general solution of the system is

$$\mathbf{y}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-3t} + c_3 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^{-3t}$$

Problem 3.3 #14: The general solution of the system is

$$\mathbf{y}(t) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}$$

Matching with the initial condition $\mathbf{y}(0) = (0,2)^t$, one can find $c_1 = c_2 = 1$.

Problem 3.4 #8: Since the underlying matrix has two real eigenvalues of opposite signs: $\lambda_1 = 2$ and $\lambda_2 = -5$, the critical point $y_1 = y_2 = 0$ is a saddle point. The real general solution is

$$\mathbf{y}(t) = c_1 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t}$$

Problem 3.5 #8: There are three critical points at $y = 0$ (center), $y = 1$ (saddle point), and $y = -1$ (saddle point). The nonlinear equation can be linearized either as the second-order scalar equation or as a system of two differential equations.

Problem 18.8 #6: The power method with scaling predicts the following values for the dominant eigenvalue: $\lambda^{(1)} = 9.93$, $\lambda^{(2)} = 12.37$, $\lambda^{(3)} = 10.52$, $\lambda^{(4)} = 10.52$, $\lambda^{(5)} = 11.77$, $\lambda^{(6)} = 10.86$, $\lambda^{(7)} = 11.50$, $\lambda^{(8)} = 11.05$, $\lambda^{(9)} = 11.36$, $\lambda^{(10)} = 11.14$. The convergence is very slow. The scaled eigenvector for the dominant eigenvalue after 3,5,10 iterations is

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0.49 \\ 1 \\ 0.58 \\ 0.79 \end{pmatrix}, \quad \mathbf{x}^{(5)} = \begin{pmatrix} 0.51 \\ 1 \\ 0.58 \\ 0.83 \end{pmatrix}, \quad \mathbf{x}^{(10)} = \begin{pmatrix} 0.53 \\ 1 \\ 0.60 \\ 0.87 \end{pmatrix}$$

Problem 18.9 #8: After 3,5 iterations of the QR factorization algorithm, the matrix A is

$$A^{(3)} = \begin{pmatrix} 14.20 & 0.01 & 0 \\ 0.01 & -6.31 & 0.007 \\ 0 & 0.007 & 2.105 \end{pmatrix}, \quad A^{(5)} = \begin{pmatrix} 14.20 & 0.002 & 0 \\ 0.002 & -6.305 & 0.0008 \\ 0 & 0.0008 & 2.1048 \end{pmatrix}.$$

After 10 iterations, the matrix $A^{(10)}$ is diagonal with the following approximation for eigenvalues: $\lambda_1 \approx 14.2$, $\lambda_2 \approx -6.305$, and $\lambda_3 \approx 2.105$.