Completeness relations in L^2_{per} space for the BO and NDNLS$^\pm$ equations using eigenfunctions of the Lax operator

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The NDNLS$^{\pm}$ and BO equations

The Benjamin-Ono equation in normalized form is given by

$$u_t + 2uu_x + H(u_{xx}) = 0, \quad u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

and the nonlocal derivative nonlinear Schrödinger equation can be written as

$$iu_t = u_{xx} + \sigma u(i + H)(|u|^2)_x, \quad u(x,t) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}.$$

The BO equation (Benjamin 1967, Ono 1975) and the NDNLS$^+$ equation (Pelinovsky and Grimshaw 1995) can be used to describe physical phenomena such as deep water waves and internal waves in a stratified fluid. The NDNLS$^-$ equation can be used to model the continuous limit of the Calogero-Moser-Sutherland particle system (Abanov et al. 2009).

Periodic travelling waves

The BO and NDNLS$^{\pm}$ equations admit periodic travelling wave solutions, which can be written in Hirota bilinear form as

$$u(x,t) = i\partial_x \log\frac{f}{f'} = i\left(\frac{f_x}{f} - \frac{f'_x}{f'}\right), \quad \begin{cases} f = 1 + e^{ik\xi-\phi} \\ f' = 1 + e^{ik\xi+\phi} \end{cases}.$$

for the BO equation, and

$$u(x-ct) = e^{i\theta}\frac{g(x-ct)}{f(x-ct)}, \quad \overline{u}(x-ct) = e^{-i\theta}\frac{g'(x-ct)}{f'(x-ct)},$$

$$\begin{cases} f(x-ct) = 1 + e^{ik\xi-\phi}, \\ f'(x-ct) = 1 + e^{ik\xi+\phi}, \end{cases} \qquad \begin{cases} g(x-ct) = \gamma(1 + e^{ik\xi-\psi}), \\ g'(x-ct) = \gamma^{-1}(1 + e^{ik\xi+\psi}), \end{cases}$$

for the NDNLS$^{\pm}$ equation.

The NDNLS$^{\pm}$ and BO equations

Solitons in the BO and NDNLS^{+} equations have been studied previously (Matsuno and Kaup 1997, Matsuno 2001). For both equations, a completeness relation for eigenfunctions linearized at the soliton solution was established and used to investigate linear stability with respect to small perturbations.

We aim to derive a completeness relation in L^2_{per} space for eigenfunctions of the BO equation linearized at the periodic travelling wave using the Hirota bilinear form, based on previous work (Matsuno and Kaup 1997, Spector and Miloh 1994). We will apply these results to perform the same task for the periodic travelling wave solution of the NDNLS$^{\pm}$ equation.

What is a completeness relation?

A completeness relation in a Hilbert space $\mathcal{H}$ means that any vector in the space can be represented as a linear combination of elements of a basis.

Example

The Fourier basis $\{e^{inx}\}_{n\in\mathbb{Z}}$ forms a basis for the Hilbert space L^2_{per} in the sense that any $f(x) \in L^2_{per}$ can be represented by

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}, \quad a_n = \langle f, e^{inx} \rangle.$$

The equality is in the sense of L^2_{per} convergence, i.e.

$$\lim_{N\to\infty} \left\| f - \sum_{n=-N}^{N} a_n e^{inx} \right\|_{L^2_{per}} = 0.$$

Applications

We can use a completeness relation to decompose general elements of our Hilbert space into known functions.

Definition (Linear Stability)

If we perturb a solution u to a partial differential equation by a small perturbation v, expand, and drop higher order terms in v, we get a linear equation in v. If the norm of v does not grow over time, we say that the solution u is **linearly stable**.

Linear stability of the BO constant solution

We can use the Fourier basis $\{e^{inx}\}_{n\in\mathbb{Z}}$ to examine the linear stability of the constant solution of the BO equation in L^2_{per}.

Example

Take the solution $\tilde{u} = 1$ of the BO equation. Taking a small perturbation $v(x,t) \in L^2_{per}$, we perturb $\tilde{u}$ by v to get $u(x,t) = 1 + v(x,t)$. Substituting u into the BO equation, we obtain

$$v_t + 2(1+v)v_x + Hv_{xx} = 0.$$

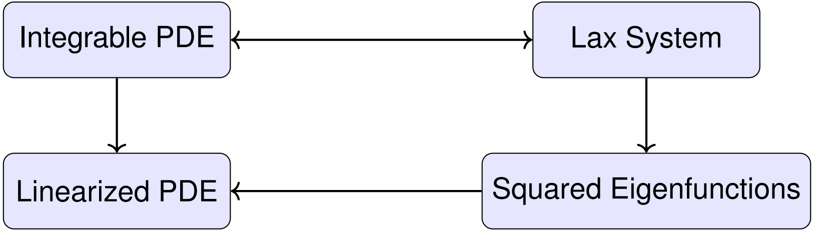
We can make a change of variables and linearize to get $v_t + Hv_{xx} = 0$. Using our Fourier basis, we can write $v(x,t) = \sum_{n\in\mathbb{Z}} v_n(t)e^{inx}$. Plugging this into the linearized equation, we can solve for $v_n(t)$:

$$v_n(t) = v_n(0)e^{in|n|t} \quad \implies \quad \|v_n(t)\|_{L^2_{per}} = \|v_n(0)\|_{L^2_{per}}.$$

Therefore, the constant solution is neutrally stable.

Lax system and spectrum

Recent work has rigorously established the Lax spectrum for the periodic travelling wave solution of the BO equation (Gérard and Kappellar 2020, 2021, 2022). Additionally, the Lax spectrum and solutions of the Lax system for the periodic travelling wave of the NDNLS$^{\pm}$ equation have been recently established (Chen and Pelinovsky 2025).



BO: Linearized equations

We can perturb the periodic travelling wave u by some small perturbation $q \in L^2_{per}$. Substitution and linearization yields the **linearized BO equation**,

$$q_t + 2(uq)_x + H(q_{xx}) = 0.$$

Defining $q = \psi_x$, we can integrate to obtain the **adjoint linearized BO equation**,

$$\psi_t + 2u\psi_x + H(\psi_{xx}) = 0.$$

BO: Lax spectrum

The BO equation is a compatibility condition of the following system of Lax equations

$$\begin{cases} i\varphi_x^+ + \lambda(\varphi^+ - \varphi^-) + u\varphi^+ = 0, \\ i\varphi_t^\pm - 2i\lambda\varphi_x^\pm + \varphi_{xx}^\pm - [\pm i u_x + H(u_x)]\varphi^\pm = 0. \end{cases}$$

The spectrum of the Lax system in L^2_{per} for the periodic travelling wave is given by

$$\sigma_L = \{\lambda_0\} \cup \{0\} \cup \{kn\}_{n\in\mathbb{N}}, \quad \lambda_0 = -\frac{c+k}{2}, \quad \mu_0 = -\frac{c-k}{2}$$

This system admits two sets of eigenfunctions

$$\varphi_I^+(\lambda; x, t) = e^{i\lambda x + i\lambda^2 t}\frac{f}{f'}, \quad \varphi_I^- = 0, \quad \lambda \in [0, \infty),$$

$$\varphi_{II}^+(\lambda; x, t) = 1 - \frac{k}{(\lambda - \lambda_0)f}, \quad \varphi_{II}^- = 1 - \frac{k}{(\lambda - \lambda_0)f'}, \quad \lambda \in \mathbb{R} \setminus \{\lambda_0\}.$$

BO: Linearized equation solutions

The linearized BO equation admits the following solutions:

$$\begin{cases} q_I(\lambda; x, t) = \dfrac{\lambda - \lambda_0}{\lambda - \mu_0}\, \varphi_I^+(\lambda; x, t)\overline{\varphi_{II}^-}(\lambda; x, t), \\ q_{II}(\lambda; x, t) = e^{-2\phi}\, \overline{\varphi_I^+}(\lambda; x, t)\varphi_{II}^-(\lambda; x, t), \end{cases} \qquad \lambda \in [0, \infty).$$

The BO equation has translational symmetries in x and t. A result of these symmetries are that the following derivatives satisfy the linearized equation:

$$q_1 = \partial_x u(x - ct), \quad \partial_c u(x - ct).$$

The second solution has a term proportional to $\partial_x u(x - ct)$, we can drop this to obtain $q_2 = \partial_\phi u(x - ct)$.

BO: Adjoint linearized equation solutions

The adjoint linearized BO equation admits the following solutions:

$$\begin{cases} \psi_I(\lambda; x, t) = \dfrac{\lambda - \lambda_0}{\lambda - \mu_0}\, \varphi_I^+(\lambda; x, t)\overline{\varphi_{II}^+}(\lambda; x, t), \\ \psi_{II}(\lambda; x, t) = e^{-2\phi}\, \overline{\varphi_I^+}(\lambda; x, t)\varphi_{II}^+(\lambda; x, t), \end{cases} \qquad \lambda \in [0, \infty).$$

Using the relation $\psi = \partial_x^{-1} q$, we obtain the solutions

$$\psi_1 = u(x - ct), \quad \partial_x^{-1}\partial_c u(x - ct).$$

Again, we can omit the term proportional to $u(x - ct)$ in the second solution to obtain $\psi_2 = \partial_x^{-1}\partial_\phi u(x - ct)$.

BO: Orthogonality relations

We examine orthogonality relations between solutions to determine coefficients for use in the completeness relation. We have the following results:

$$\begin{cases} \langle q_I(kn), \psi_I(km)\rangle_{L^2_{\mathrm{per}}} = \langle q_{II}(kn), \psi_{II}(km)\rangle_{L^2_{\mathrm{per}}} = 0, \\ \langle q_I(kn), \psi_{II}(km)\rangle_{L^2_{\mathrm{per}}} = \langle q_{II}(kn), \psi_I(km)\rangle_{L^2_{\mathrm{per}}} = \dfrac{2\pi}{k}\delta_{nm}, \end{cases} \qquad n, m \in \mathbb{N},$$

and

$$\begin{cases} \langle q_1, \psi_1\rangle_{L^2_{\mathrm{per}}} = \langle q_2, \psi_2\rangle_{L^2_{\mathrm{per}}} = 0, \\ \langle q_1, \psi_2\rangle_{L^2_{\mathrm{per}}} = -\langle q_2, \psi_1\rangle_{L^2_{\mathrm{per}}} = \dfrac{k\pi}{\sinh^2\phi}. \end{cases}$$

BO: Completeness relation

Proposition (Completeness relation in L^2_{per})

The functions $\psi_I, \psi_{II}, \psi_1$ and ψ_2 form a basis in L^2_{per} in the sense that any element $f(x) \in L^2_{per}$ can be uniquely represented by a superposition of elements in the basis:

$$f(x) = \sum_{n\in\mathbb{N}} (a_n \psi_I(kn; x) + b_n \psi_{II}(kn; x)) + c_0 + c_1\psi_1 + c_2\psi_2, \quad x \in \mathbb{T},$$

where the coefficients are given by

$$a_n = \frac{k}{2\pi}\langle q_{II}(kn), f\rangle_{L^2_{per}}, \quad b_n = \frac{k}{2\pi}\langle q_I(kn), f\rangle_{L^2_{per}}, \quad n \in \mathbb{N},$$

$$c_1 = -\frac{1}{\pi k}\sinh^2\phi\langle q_2, f\rangle_{L^2_{per}}, \quad c_2 = \frac{1}{\pi k}\sinh^2\phi\langle q_1, f\rangle_{L^2_{per}},$$

$$\text{and} \quad c_0 = \frac{k}{4\pi}\left(\langle q_{II}(0), f\rangle_{L^2_{per}}\psi_I(0) + \langle q_I(0), f\rangle_{L^2_{per}}\psi_{II}(0)\right).$$

BO: Completeness relation

Substituting our coefficients into our relation, we obtain

$$f(x) = \frac{k}{2\pi} \oint f(y) \left(\sum_{n\in\mathbb{N}} (q_{II}(kn; y)\psi_I(kn; x) + q_I(kn; y)\psi_{II}(kn; x)) \right.$$
$$+\frac{1}{2} [q_{II}(0; y)\psi_I(0; x) + q_I(0; y)\psi_{II}(0; x)]$$
$$\left. +\frac{2}{k^2} \sinh^2 \phi \, [-q_2(y)\psi_1(x) + q_1(y)\psi_2(x)] \right) dy.$$

Therefore, the completeness relation holds if and only if

$$\sum_{n\in\mathbb{N}} (q_I(kn; x)\psi_{II}(kn; y) + q_{II}(kn; x)\psi_I(kn; y)) + \frac{1}{2}(q_I(0; x)\psi_{II}(0; y) + q_{II}(0; x)\psi_I(0; y))$$
$$+ \frac{2}{k^2} \sinh^2 \phi (q_1(x)\psi_2(y) - q_2(x)\psi_1(y)) = \frac{2\pi}{k}\delta(x - y), \quad x, y \in \mathbb{T}.$$

NDNLS: Linearized equation

The **linearized NDNLS$^{\pm}$ equation** can be obtained by perturbing the periodic travelling wave u by a small perturbation $q \in L^2_{per}$, followed by substitution and linearization. This yields

$$iq_t = q_{xx} + \sigma[u(i+H)(q\bar{u} + u\bar{q})_x + q(i+H)(|u|^2)_x].$$

NDNLS: Lax spectrum

The NDNLS$^{\pm}$ equation is a compatibility condition of the following system of Lax equations:

$$\begin{cases} ip_x + \lambda p + uq^+ = 0, \\ q^+ - \mu q^- + \sigma \bar{u} p = 0, \\ ip_t + \lambda^2 p + \lambda u q^+ + i(uq_x^+ - u_x q^+) = 0, \\ iq_t^{\pm} - 2i\lambda q_x^{\pm} + q_{xx}^{\pm} + \sigma q^{\pm}[(\pm i + H)(|u|^2)_x] = 0. \end{cases}$$

When u is the periodic travelling wave, the spectrum of the Lax system in L^2_{per} is given by

$$\sigma_L = \{\lambda_0\} \cup \{\sigma + k(n-1)\}_{n \in \mathbb{N}}.$$

NDNLS: Lax Spectrum

This system admits two sets of eigenfunctions, given by

$$\begin{cases} p_I(\lambda, x, t) = e^{i(\lambda-\sigma)x+i(\lambda^2-1)t}\dfrac{f'(x-ct)}{f(x-ct)}, & \\ q_I^+(\lambda; x, t) = -\sigma e^{i(\lambda-\sigma)x+i(\lambda^2-1)t}\dfrac{g'(x-ct)}{f(x-ct)}, & \lambda \in [\sigma, \infty) \\ q_I^- = 0, & \end{cases}$$

and

$$\begin{cases} p_{II}(\lambda; x, t) = -\dfrac{1}{\lambda f(x-ct)}\left(\gamma + \dfrac{\lambda-\lambda_0}{\lambda-\mu_0}(g(x-ct)-\gamma)\right), & \\ q_{II}^+(\lambda; x, t) = \dfrac{1}{f(x-ct)}\left(1 + \dfrac{\lambda-\lambda_0}{\lambda-\mu_0}(f(x-ct)-1)\right), & \lambda \in \mathbb{R} \setminus \{\mu_0\} \\ q_{II}^-(\lambda; x, t) = \dfrac{1}{f'(x-ct)}\left(1 + \dfrac{\lambda-\lambda_0}{\lambda-\mu_0}(f'(x-ct)-1)\right). & \end{cases}$$

NDNLS: Linearized equation solutions

The linearized NDNLS$^{\pm}$ equation admits the following solution:

$$w(\lambda; x, t) = p_I(\lambda; x, t)\overline{q_{II}^{-}}(\lambda; x, t), \quad \lambda \in [\sigma, \infty).$$

The NDNLS$^{\pm}$ equation has translational symmetries in x and t, in addition to a phase rotation symmetry. Thus, there are three corresponding solutions of the linearized NDNLS$^{\pm}$ equation, given by

$$h_1 = \partial_x u(x - ct), \quad \partial_c u(x - ct), \quad \text{and} \quad h_3 = \partial_\theta u(x - ct).$$

One term in $\partial_c u(x - ct)$ is proportional to $\partial_x u(x - ct)$; we can drop this term for use in the completeness relation. We define the new function as h_2.

NDNLS: Orthogonality relations

Based on previous work (Matsuno 2001), we propose the following solution of the adjoint linearized NDNLS$^{\pm}$ equation:

$$\tilde{w}(\lambda; x, t) = \overline{p_I}(\lambda; x, t) q_{II}^{+}(\lambda; x, t), \quad \lambda \in [\sigma, \infty).$$

There also must be three solutions of the adjoint linearized NDNLS$^{\pm}$ equation corresponding to the functions h_1, h_2, and h_3; we will denote these by $\tilde{h}_1$, $\tilde{h}_2$, and $\tilde{h}_3$.

We have the following orthogonality relation between the solution w and the proposed solution $\tilde{w}$:

$$\langle w(kn), \tilde{w}(km) \rangle_{L^2_{per}} = \frac{2\pi}{k} \delta_{nm}, \quad n, m \in \mathbb{Z}, \; n, m \geq \sigma.$$

Completeness relation conjecture

Conjecture (Completeness relation in L^2_{per})

We conjecture that the functions w, h_1, h_2, and h_3 form a basis in L^2_{per} in the sense that any element $f(x) \in L^2_{per}$ can be uniquely represented by a superposition of elements in the basis:

$$f(x) = \sum_{n=\sigma}^{\infty} a_n w(kn; x, t) + c_1 h_1 + c_2 h_2 + c_3 h_3, \quad x \in \mathbb{T},$$

where the coefficient $a_n = \frac{k}{2\pi}\langle \tilde{w}(kn), f\rangle$, and the coefficients c_1, c_2, and c_3 will be determined from the orthogonality relations between h_1, h_2, h_3 and the corresponding functions $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$.

Next steps

In order to prove a completeness relation in L^2_{per}, it remains to

1. Find the remaining adjoint eigenfunctions corresponding to the discrete spectrum,
2. Show the proposed solution $\tilde{w}$ and the adjoint eigenfunctions for the discrete spectrum satisfy the adjoint linearized NDNLS$^{\pm}$ equation,
3. Find orthogonality relations between adjoint and linearized eigenfunctions for the discrete spectrum, and
4. Prove a completeness relation with the appropriate eigenfunctions and coefficients.

Additionally, we aim to establish a completeness relation in the space $L^2(\mathbb{R})$. We have found a completeness relation for the BO equation in $L^2(\mathbb{R})$, but it remains to apply these results to the NDNLS$^{\pm}$ equation.

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