Completeness relations in L^2 spaces for the periodic travelling wave of the BO and NDNLS equations using eigenfunctions of the Lax operator

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1 Introduction

We consider the periodic travelling wave solutions of the Benjamin-Ono (BO) equation and the focusing and defocusing nonlocal derivative nonlinear Schrödinger (NDNLS±) equations. We aim to establish completeness relations in the spaces of periodic square-integrable functions and square-integrable functions over $\mathbb{R}$ for eigenfunctions of these equations linearized about their respective periodic travelling wave solutions. Completeness relations in the appropriate spaces for the NDNLS$^{\pm}$ equation can be applied to examine the linear stability of the periodic travelling wave solution with respect to perturbations in L^2_{per} or $L^2(\mathbb{R})$.

1.1 The BO and NDNLS$^{\pm}$ equations

We consider the BO equation in normalized form

$$u_t + 2uu_x + H(u_{xx}) = 0, \quad u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \tag{1.1}$$

and the NDNLS$^{\pm}$ equation

$$iu_t = u_{xx} + \sigma u(i + H)(|u|^2)_x, \quad u(x,t) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}, \tag{1.2}$$

where H is the Hilbert transform and $\sigma = \pm 1$ is a sign parameter corresponding to the NDNLS$^+$ ($\sigma = +1$) or NDNLS$^-$ ($\sigma = -1$) equation. The Hilbert transform can be defined by

$$H(f(x)) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{y - x} dy,$$

or according to the Fourier transform by $H(e^{ikx}) = i\,\mathrm{sgn}(k)e^{ikx}$ for $k \in \mathbb{R}$. Additionally, for functions $f^{\pm}$ analytic in the upper or lower half complex plane, the Hilbert transform has the property that $Hf^{\pm} = \pm i f^{\pm}$.

The BO and NDNLS$^{\pm}$ equations are derived from physical contexts. The BO equation describes unidirectional propagation of internal waves in a deep stratified fluid [2, 15], and the NDNLS^{+} equation was derived from the BO equation in the context of modulation theory relating to deep fluids [16]. The NDNLS^{-} equation was derived more recently in the context of the continuous limit of the Calogero-Moser-Sutherland system of particles [1].
A completeness relation for eigenfunctions of the BO equation linearized about multisoliton solutions was established in [12, 13] using the Hirota bilinear form, and was used to investigate the linear stability of the multisoliton solution with respect to small perturbations. Following this, the same methods were used to construct a completeness relation for eigenfunctions of the NDNLS^{+} equation linearized about multisoliton solutions and to analyze the linear stability of the multisoliton solution [11]. Additionally, the linear stability of the periodic travelling wave in the BO equation was analyzed and a completeness relation was established in [17] using the inverse scattering transform.
The spectrum of the Lax operator for the periodic travelling wave solution of the BO equation has been recently studied and rigorously justified [7, 8, 9]. Following this work, the Lax spectrum and eigenfunctions for the periodic travelling wave solution of the NDNLS$^{\pm}$ equation were established in [3]. The recent advances in theory regarding the periodic travelling wave solutions of these equations have motivated the present work.

In this work, we establish completeness relations in L^2_{per} and $L^2(\mathbb{R})$ for eigenfunctions of the BO equation linearized about the periodic travelling wave solution. The completeness relations are constructed using solutions of the linearized BO and adjoint linearized BO equations obtained from eigenfunctions of the Lax system. This work is then applied to the NDNLS$^{\pm}$ equation. We use the Hirota bilinear form of the periodic travelling wave to express solutions of the linearized NDNLS$^{\pm}$ and adjoint linearized NDNLS$^{\pm}$ equations in terms of eigenfunctions of the Lax operator and conjecture a completeness relation in L^2_{per}. The Hirota bilinear form of both periodic travelling wave solutions is used throughout for clarity.

1.2 What is a completeness relation?

A completeness relation in a Hilbert space $\mathcal{H}$ means that any vector in the space can be represented as a linear combination of elements of a basis. For example, the n-dimensional Euclidean space $\mathbb{R}^n$ is a Hilbert space with standard inner product defined by the dot product of two finite dimensional vectors. One basis for $\mathbb{R}^n$ is given by the standard basis vectors $\{e_1, \ldots, e_n\}$; this is an orthonormal basis. This basis forms a completeness relation for $\mathbb{R}^n$ in the sense that any vector $x \in \mathbb{R}^n$ can be represented as a linear combination of the basis vectors $\{e_1, \dots, e_n\}$ as

$$x = x_1 e_1 + \cdots + x_n e_n$$

where the coefficient x_i is the ith coordinate of x, given by $x_i = e_i \cdot x$.
We are considering the infinite dimensional Hilbert spaces $L^2(\mathbb{R})$ and L^2_{per} where the inner product of two functions f, g in L^2 space is typically defined by

$$\langle f(x), g(x) \rangle = \int f(x) \overline{g(x)} dx,$$

where integration is performed over $\mathbb{R}$ in $L^2(\mathbb{R})$ or the period of functions in L^2_{per}. Since these spaces are infinite dimensional, a basis for elements in these spaces must be generalized from a finite set of elements to a countable or uncountable set of elements.
For example, a well known basis for L^2_{per} is given by the Fourier basis $\{e^{inx}\}_{n\in\mathbb{Z}}$. Any function $f(x) \in L^2_{per}$ with period T can be represented by the sum

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}, \quad a_n = \frac{1}{T} \langle f, e^{inx} \rangle.$$

Since this is an infinite sum, the equality is in terms of convergence of the L^2 norm of the infinite sum to f:

$$\lim_{N\to\infty} \left\| f - \sum_{n=-N}^{N} a_n e^{inx} \right\|_{L^2_{per}} = 0.$$

This equality can be shown using the fact that $|e^{inx}|^2 = 1$ for all $n \in \mathbb{Z}$ and using Parseval's equality to obtain $\oint |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |a_n|^2$.
For $L^2(\mathbb{R})$, a completeness relation can be constructed from an uncountably large basis in the sense of the Dirac delta distribution. To demonstrate this in an example, take linearly independent functions $\phi_1(x;\lambda), \phi_2(x;\lambda)$ and $\psi_1(x;\lambda), \psi_2(x;\lambda)$ dependent on the parameter λ with appropriate orthogonality relations such that there are coefficients $a_1(\lambda) = \langle f(y), \psi_1(y;\lambda) \rangle$ and $a_2(\lambda) = \langle f(y), \psi_2(y;\lambda) \rangle$ for $f \in L^2(\mathbb{R})$. Then we have the following completeness relation

$$f(x) = \int_{\mathbb{R}} [(a_1(\lambda)\phi_1(x;\lambda) + a_2(\lambda)\phi_2(x;\lambda)] d\lambda,$$

where equality is in terms of the $L^2(\mathbb{R})$ norm. After expanding the coefficients and interchanging the order of integration, we get

$$f(x) = \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} [(\psi_1(y;\lambda)\phi_1(x;\lambda) + \psi_2(y;\lambda)\phi_2(x;\lambda)] d\lambda \right) dy,$$

which holds if and only if

$$\int_{\mathbb{R}} [(\psi_1(y;\lambda)\phi_1(x;\lambda) + \psi_2(y;\lambda)\phi_2(x;\lambda)] d\lambda = \delta(x-y).$$

This example involves the uncountably infinite set of basis functions $\{\phi_1(x;\lambda), \phi_2(x;\lambda)\}$ for $\lambda \in \mathbb{R}$. The completeness relation is satisfied if the 'uncountably infinite sum' of basis functions represented by the integral over $\mathbb{R}$ converges in the sense of distributions to $f(x)$.

1.3 An example: Completeness relation applied to linear stability analysis

A completeness relation allows us to decompose a general element into a linear combination of basis elements. This can be useful because we may have more information about the basis elements than we do about the general element. An example of an application of a completeness relation in a function space is the analysis of the linear stability of solutions of a partial differential equation.

Definition 1.1. *If we perturb a solution u to a partial differential equation by a small perturbation $v \in \mathcal{H}$ for the relevant Hilbert space $\mathcal{H}$, we can substitute the perturbed solution into our partial differential equation. After expanding and dropping nonlinear terms in v, we are left with a linear equation in v. Solving for v, if the norm in $\mathcal{H}$ does not grow over time, we say u is **linearly stable** with respect to perturbations in $\mathcal{H}$.*

To demonstrate how a completeness relation can be applied to examine the linear stability of a solution, I will show how the Fourier basis $\{e^{inx}\}_{n\in\mathbb{Z}}$ can be applied to show that the constant solution of the BO equation (1.1) is linearly stable with respect to perturbations in the space L^2_{per}.

Consider the constant solution $\tilde{u}(x,t) = C \in \mathbb{R}$ of the BO equation. We can consider a perturbation $\tilde{v}(x,t) \in L^2_{per}$ and perturb $\tilde{u}$ by $\tilde{v}$ to get $u(x,t) = C + \tilde{v}(x,t)$. By making the change of variables $x \mapsto x - 2Ct$, we have $u(x-2Ct,t) = 1 + v(x-2Ct,t)$. Substituting into the BO equation and linearizing with respect to v, we have

$$v_t + H(v_{xx}) = 0.$$

We can make change variables back for easier notation: $x - 2Ct \mapsto x$. Since $v \in L^2_{per}$ with respect to x, we can apply our completeness relation using the Fourier basis to obtain

$$v(x,t) = \sum_{n\in\mathbb{Z}} v_n(t)e^{inx}.$$

Thus, the time dependence of v is represented by $v_n(t)$ for each n. We are interested in the time dependence of the L^2_{per} norm of v. Substituting into our linearized equation, we can solve to yield $v_n(t) = v_n(0)e^{in|n|t}$ for each n. Hence, the time dependence for each n is dependent on a complex exponential, which is cancelled in the L^2_{per} norm. This is true for each n, so the L^2_{per} norm of the perturbation v does not grow over time. Therefore, the constant solution C of the BO equation is linearly stable.

The Fourier basis can also be used to determine that the constant solution of the NDNLS$^{\pm}$ equation is linearly stable. However, different completeness relations for perturbations in L^2_{per} and $L^2(\mathbb{R})$ must be used to determine the linear stability of nontrivial solutions such as the periodic travelling wave solutions.

2 The Benjamin-Ono equation

We consider the travelling periodic wave solution of the BO equation (1.1). We construct solutions to the linearized BO and adjoint linearized BO equations using eigenfunctions of the Lax operator, where linearization occurs at the travelling periodic wave solution. Additional solutions are constructed by exploiting symmetries of the BO equation. The functions obtained from these solutions are used in completeness relations for the function spaces L^2_{per} and $L^2(\mathbb{R})$ with coefficients obtained from orthogonality relations.

2.1 The periodic travelling wave solution of the BO equation

We consider the BO equation in the normalized form (1.1). The BO equation is a compatibility condition of the following system of Lax equations [6, 14]:

$$\begin{cases} i\varphi^+_x + \lambda(\varphi^+ - \varphi^-) + u\varphi^+ = 0, \\ i\varphi^\pm_t - 2i\lambda\varphi^\pm_x + \varphi^\pm_{xx} - [\pm iu_x + H(u_x)]\varphi^\pm = 0, \end{cases} \tag{2.1}$$

where $\varphi^\pm$ are analytic in $\mathbb{C}^\pm$.
The BO equation has two physical translational symmetries in x and t. Given a solution $u(x,t)$, the function $u(x-ct+\xi_0)$ is a solution for $c, \xi_0 \in \mathbb{R}$. Without loss of generality, we can set $\xi_0 = 0$ and consider the solution $u(\xi)$ where $\xi = x - ct$ is the travelling wave coordinate for wave speed c.

The BO equation (1.1) admits a periodic travelling wave solution which can be expressed in terms of elementary functions as

$$u(x,t) = \frac{k\sinh\phi}{\cos(k\xi) + \cosh\phi}, \tag{2.2}$$

where $\xi = x - ct - \xi_0$ is the travelling wave coordinate, $k, \phi > 0$ are arbitrary parameters, and the wave speed c is given by $c = k\coth\phi$. We can set $\xi_0 = 0$ without loss of generality.
To perform analysis of the periodic travelling wave (2.2), we use the Hirota bilinear form

$$u = i\partial_x \log\frac{f}{f'} = i\left(\frac{f_x}{f} - \frac{f'_x}{f'}\right), \quad \begin{cases} f = 1 + e^{ik\xi - \phi}, \\ f' = 1 + e^{ik\xi + \phi} \end{cases}. \tag{2.3}$$

We observe that because both $k, \phi > 0$, $\frac{1}{f}$ is analytic in $\mathbb{C}^+$ and $\frac{1}{f'}$ is analytic in $\mathbb{C}^-$.
The Lax spectrum of the periodic travelling wave (2.2) was recently rigorously justified in [7]. The Lax operator is self-adjoint in the Hilbert spaces of $L^2(\mathbb{R})$ and L^2_{per} and so from spectral theory we know that the spectrum of the Lax operator in both of these function spaces will be a subset of $\mathbb{R}$. For eigenfunctions of the Lax system (2.1) in $L^2(\mathbb{R})$, the Lax spectrum of the periodic travelling

wave (2.2) is given by

$$\sigma_L|_{L^2(\mathbb{R})} = [\lambda_0, \mu_0] \cup [0, \infty), \quad \lambda_0 = -\frac{c+k}{2}, \quad \mu_0 = -\frac{c-k}{2}, \tag{2.4}$$

and for eigenfunctions in L^2_{per}, it is given by

$$\sigma_L|_{L^2_{per}} = \{\lambda_0\} \cup \{0\} \cup \{kn\}_{n\in\mathbb{N}}, \tag{2.5}$$

see Proposition C.2 and Proposition 2.2 of [7]. We use some useful relations between the parameters k, ϕ and λ_0, μ_0 by exploiting $c = k \coth \phi$:

$$k = \mu_0 - \lambda_0, \quad e^{2\phi} = \frac{\lambda_0}{\mu_0}. \tag{2.6}$$

To study the linearized BO equation, we consider a perturbation q to the periodic travelling wave u (2.2), which yields

$$q_t + 2(uq)_x + H(q_{xx}) = 0. \tag{2.7}$$

Defining $\psi = \partial_x^{-1} q$, we get the adjoint linearized BO equation:

$$\psi_t + 2u\psi_x + H(\psi_{xx}) = 0. \tag{2.8}$$

2.2 Eigenfunctions

When u is the travelling periodic wave (2.3), the Lax system (2.1) admits two sets of eigenfunctions [4, 5, 8, 9]

$$\varphi_I^+(\lambda; x, t) = e^{i\lambda x + i\lambda^2 t} \frac{f'(x-ct)}{f(x-ct)}, \quad \varphi_I^-(\lambda; x, t) = 0, \quad \lambda \in [0, \infty) \tag{2.9}$$

and

$$\varphi_{II}^+(\lambda; x, t) = 1 - \frac{k}{(\lambda-\lambda_0)f(x-ct)}, \quad \varphi_{II}^-(\lambda; x, t) = 1 - \frac{k}{(\lambda-\lambda_0)f'(x-ct)}, \quad \lambda \in \mathbb{R}\setminus\{\lambda_0\}. \tag{2.10}$$

Based on the computations in [12, 13, 17], we have derived four solutions to the adjoint linearized BO equation (2.8) from the sets of eigenfunctions (2.9), (2.10).

Proposition 2.1. *The adjoint linearized BO equation* (2.8) *admits the following two sets of solutions:*

$$\psi_I(\lambda; x, t) = e^{i\lambda x + i\lambda^2 t} \frac{f'(x-ct)}{f(x-ct)} \left[1 + \frac{k}{(\lambda-\mu_0)f'(x-ct)}\right], \tag{2.11}$$

$$\psi_{II}(\lambda; x, t) = e^{-i\lambda x - i\lambda^2 t} \frac{f(x-ct)}{f'(x-ct)} \left[1 - \frac{k}{(\lambda-\lambda_0)f(x-ct)}\right] \tag{2.12}$$

for $\lambda \in [0, \infty)$*, and*

$$\psi_{III}(\lambda; x, t) = \frac{e^{-i(\lambda_0 x + \lambda_0^2 t)}}{f'(x - ct)} \psi_I(\lambda; x, t) \tag{2.13}$$

$$\psi_{IV}(\lambda; x, t) = \frac{e^{i(\mu_0 x + \mu_0^2 t)}}{f(x - ct)} \psi_{II}(\lambda; x, t) \tag{2.14}$$

for $\lambda \in (\lambda_0, \mu_0)$.

Remark 2.2. *We can express the solutions* (2.11) *to* (2.14) *of the adjoint linearized BO equation* (2.8) *in terms of the eigenfunctions* (2.9) *and* (2.10) *of the Lax system* (2.1):

$$\psi_I(\lambda) = \frac{\lambda - \lambda_0}{\lambda - \mu_0} \varphi_I^+(\lambda) \overline{\varphi_{II}^+}(\lambda), \tag{2.15}$$

$$\psi_{II}(\lambda) = e^{-2\phi} \overline{\varphi_I^+}(\lambda) \varphi_{II}^+(\lambda), \tag{2.16}$$

for $\lambda \in [0, \infty)$*, and*

$$\psi_{III}(\lambda) = -\frac{\lambda - \lambda_0}{k e^{2\phi}} \varphi_I^+(\lambda) \overline{\varphi_{II}^+}(\lambda) \lim_{\lambda \to \lambda_0} [(\lambda - \lambda_0) \overline{\varphi_I^+}(\lambda) \varphi_{II}^+(\lambda)], \tag{2.17}$$

$$\psi_{IV}(\lambda) = -\frac{\lambda - \lambda_0}{k e^{2\phi}} \overline{\varphi_I^+}(\lambda) \varphi_{II}^+(\lambda) \lim_{\lambda \to \mu_0} [(\lambda - \lambda_0) \varphi_I^+(\lambda) \overline{\varphi_{II}^+}(\lambda)], \tag{2.18}$$

for $\lambda \in (\lambda_0, \mu_0)$. *It is important to note that the eigenfunctions of the Lax spectrum* (2.9) *are used outside of their validity of* $\lambda \in [0, \infty)$ *in* (2.17) *and* (2.18).

We can find the four corresponding solutions for the linearized BO equation (2.7) using the relation $q = \psi_x$ for each solution (2.11) to (2.14) of the adjoint linearized BO equation (2.8). The following proposition gives a simplified form of these solutions with normalization constants $\pm \frac{i}{\lambda}$.

Proposition 2.3. *The linearized BO equation* (2.7) *admits the following two sets of solutions:*

$$q_I(\lambda; x, t) = e^{i\lambda x + i\lambda^2 t} \frac{f'(x - ct)}{f(x - ct)} \left[1 + \frac{k}{(\lambda - \mu_0) f(x - ct)} \right], \tag{2.19}$$

$$q_{II}(\lambda; x, t) = e^{-i\lambda x - i\lambda^2 t} \frac{f(x - ct)}{f'(x - ct)} \left[1 - \frac{k}{(\lambda - \lambda_0) f'(x - ct)} \right], \tag{2.20}$$

for $\lambda \in [0, \infty)$*, and*

$$\begin{aligned} q_{III}(\lambda; x, t) = {} & e^{i(\lambda - \lambda_0) x + i(\lambda^2 - \lambda_0^2) t} \frac{1}{f(x - ct)} \left[1 + \frac{k}{(\lambda - \mu_0) f(x - ct)} \right] \\ & - \frac{\lambda_0}{\lambda} e^{i(\lambda - \lambda_0) x + i(\lambda^2 - \lambda_0^2) t} \frac{1}{f'(x - ct)} \left[1 + \frac{k}{(\lambda - \mu_0) f'(x - ct)} \right], \end{aligned} \tag{2.21}$$

$$q_{IV}(\lambda;x,t) = e^{-i(\lambda-\mu_0)x - i(\lambda^2-\mu_0^2)t} \frac{1}{f'(x-ct)} \left[1 - \frac{k}{(\lambda-\lambda_0)f'(x-ct)}\right]$$
$$- \frac{\mu_0}{\lambda} e^{-i(\lambda-\mu_0)x - i(\lambda^2-\mu_0^2)t} \frac{1}{f(x-ct)} \left[1 - \frac{k}{(\lambda-\lambda_0)f(x-ct)}\right], \qquad (2.22)$$

for $\lambda \in (\lambda_0, \mu_0)$.

Remark 2.4. *Once again, we can express* (2.19) *to* (2.22) *in terms of the eigenfunctions* (2.9) *and* (2.10) *of the Lax system* (2.1)*:*

$$q_I = \frac{\lambda - \lambda_0}{\lambda - \mu_0} \varphi_I^+ \overline{\varphi_{II}^-}, \qquad (2.23)$$
$$q_{II} = e^{-2\phi} \overline{\varphi_I^+} \varphi_{II}^-, \qquad (2.24)$$

for $\lambda \in [0, \infty)$*, and*

$$q_{III} = \frac{\lambda - \lambda_0}{ke^{2\phi}(\lambda - \mu_0)} \varphi_I^+ \left(\frac{\lambda_0}{\lambda} \overline{\varphi_{II}^+} \lim_{\lambda \to \lambda_0} [(\lambda - \lambda_0) \overline{\varphi_I^+} \varphi_{II}^-] - \overline{\varphi_{II}^-} \lim_{\lambda \to \lambda_0} [(\lambda - \lambda_0) \overline{\varphi_I^+} \varphi_{II}^+] \right) \qquad (2.25)$$
$$q_{IV} = \frac{1}{ke^{2\phi}} \overline{\varphi_I^+} \left(\varphi_{II}^- \lim_{\lambda \to \mu_0} [(\lambda - \lambda_0) \varphi_I^+ \overline{\varphi_{II}^+}] - \frac{\mu_0}{\lambda} \varphi_{II}^+ \lim_{\lambda \to \mu_0} [(\lambda - \lambda_0) \varphi_I^+ \overline{\varphi_{II}^-}] \right) \qquad (2.26)$$

for $\lambda \in (\lambda_0, \mu_0)$*, where all functions are a function of* λ*. Again, we are using the eigenfunctions* (2.9) *outside of their valid range in* (2.25) *and* (2.26)*.*

The eigenfunctions (2.11) to (2.14) and (2.19) to (2.22) are sufficient to continue with the proofs of orthogonality and completeness for functions in $L^2(\mathbb{R})$. However, for functions in L^2_{per}, we must examine the cases when $\lambda = 0$ and $\lambda = \lambda_0$.
When $\lambda = 0$, we have from equations (2.11) and (2.12) that

$$\psi_I(\lambda = 0) = \frac{\lambda_0}{\mu_0}, \quad \psi_{II}(\lambda = 0) = \frac{\mu_0}{\lambda_0}, \qquad (2.27)$$

so the two adjoint linearized solutions are linearly dependent and constant proportional to 1. The solutions (2.19) and (2.20) to the linearized equation (2.7) in the limit $\lambda \to 0$ are linearly independent and are given by

$$q_1(\lambda = 0) = \frac{\lambda_0}{\mu_0} \left[1 - \frac{ikf'_x}{\lambda_0\mu_0(f')^2}\right], \quad q_{II}(\lambda = 0) = \frac{\mu_0}{\lambda_0} \left[1 - \frac{ikf_x}{\lambda_0\mu_0 f^2}\right]. \qquad (2.28)$$

For $\lambda = \lambda_0$, we can use symmetries of the BO equation to find solutions to the linearized BO equation (2.7). Due to the translational symmetries of the BO equation (1.1) in x and t, the linearized BO equation (2.7) admits the solutions $\partial_x u(x - ct)$ and $\partial_c u(x - ct)$. It is important to note that both solutions are periodic. We cannot take the derivative with respect to the other parameter k, since this function will not be periodic. The solution $\partial_c u(x - ct)$ is the sum of two terms, with one term constant proportional to $\partial_x u(x-ct)$, and one term constant proportional to $\partial_\phi u(x-ct)$. The $\partial_x u(x-ct)$ term is linearly dependent with our other solution, so we can omit it for the purposes of the completeness relation. This leaves us with the functions

$$q_1 = \partial_x u(x-ct), \quad q_2 = \partial_\phi u(x-ct). \tag{2.29}$$

We can use the relation $\psi = \partial_x^{-1} q$ between the solutions of the adjoint linearized BO and linearized BO equations to obtain solutions to the adjoint linearized BO equation (2.8). This leaves us with the solutions $u(x-ct)$ and $\partial_x^{-1}\partial_c u(x-ct)$. Again, we can omit the term proportional to $u(x-ct)$ in the second solution to obtain the following functions for use in the completeness relation:

$$\psi_1 = u(x-ct), \quad \psi_2 = \partial_x^{-1}\partial_\phi u(x-ct). \tag{2.30}$$

We note that the functions q_2 and ψ_2 do not solve the linearized BO equation (2.7) or the adjoint linearized BO equation (2.8) respectively.

2.3 Orthogonality of eigenfunctions

We define orthogonality of functions in $L^2(\mathbb{R})$ by

$$f, g \in L^2(\mathbb{R}) : \langle f, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x) g(x) dx,$$

and for orthogonality of functions in L^2_{per} we define

$$f, g \in L^2_{per} : \langle f, g \rangle_{L^2_{per}} = \oint f(x) g(x) dx.$$

The integration of functions in L^2_{per} is performed over their period $\frac{2\pi}{k}$, which is independent of the starting point of integration. The typical complex conjugation seen in inner products is not used since it is included in the exponents of solutions to the adjoint linearized BO and linearized BO equations. The following propositions summarize the orthogonality relations between our solutions of equations (2.8) and (2.7) [18].

Proposition 2.5. *The set $\{\psi_I, \psi_{II}\}$ is orthogonal to the set $\{q_I, q_{II}\}$ in $L^2(\mathbb{R})$ and L^2_{per} :*

$$\begin{cases} \langle q_I(\lambda), \psi_I(\lambda') \rangle_{L^2(\mathbb{R})} = \langle q_{II}(\lambda), \psi_{II}(\lambda') \rangle_{L^2(\mathbb{R})} = 0, \\ \langle q_I(\lambda), \psi_{II}(\lambda') \rangle_{L^2(\mathbb{R})} = \langle q_{II}(\lambda), \psi_I(\lambda') \rangle_{L^2(\mathbb{R})} = 2\pi\delta(\lambda - \lambda'), \end{cases} \quad \lambda, \lambda' \in [0, \infty) \tag{2.31}$$

and

$$\begin{cases} \langle q_I(kn), \psi_I(km) \rangle_{L^2_{\text{per}}} = \langle q_{II}(kn), \psi_{II}(km) \rangle_{L^2_{\text{per}}} = 0, \\ \langle q_I(kn), \psi_{II}(km) \rangle_{L^2_{\text{per}}} = \langle q_{II}(kn), \psi_I(km) \rangle_{L^2_{\text{per}}} = \dfrac{2\pi}{k}\delta_{nm}, \end{cases} \quad n, m \in \mathbb{N} \tag{2.32}$$

where $\delta(\lambda - \lambda')$ is the Dirac delta distribution and δ_{nm} is the Kronecker delta.

Proposition 2.6. *The sets* $\{\psi_I, \psi_{II}\}$ *and* $\{q_{III}, q_{IV}\}$*, and* $\{\psi_{III}, \psi_{IV}\}$ *and* $\{q_I, q_{II}\}$ *are orthogonal in* $L^2(\mathbb{R})$ *:*

$$\begin{cases} \langle q_I(\lambda), \psi_{III}(\lambda')\rangle_{L^2(\mathbb{R})} = \langle q_{II}(\lambda), \psi_{III}(\lambda')\rangle_{L^2(\mathbb{R})} = 0, \\ \langle q_I(\lambda), \psi_{IV}(\lambda')\rangle_{L^2(\mathbb{R})} = \langle q_{II}(\lambda), \psi_{IV}(\lambda')\rangle_{L^2(\mathbb{R})} = 0, \end{cases} \quad \lambda \in (0,\infty), \quad \lambda' \in [\lambda_0, \mu_0] \tag{2.33}$$

and

$$\begin{cases} \langle q_{III}(\lambda), \psi_I(\lambda')\rangle_{L^2(\mathbb{R})} = \langle q_{IV}(\lambda), \psi_I(\lambda')\rangle_{L^2(\mathbb{R})} = 0, \\ \langle q_{III}(\lambda), \psi_{II}(\lambda')\rangle_{L^2(\mathbb{R})} = \langle q_{IV}(\lambda), \psi_{II}(\lambda')\rangle_{L^2(\mathbb{R})} = 0, \end{cases} \quad \lambda' \in (0,\infty), \quad \lambda \in [\lambda_0, \mu_0]. \tag{2.34}$$

Proposition 2.7. *The set* $\{\psi_{III}, \psi_{IV}\}$ *is orthogonal to the set* $\{q_{III}, q_{IV}\}$ *in* $L^2(\mathbb{R})$ *:*

$$\begin{cases} \langle q_{III}(\lambda), \psi_{III}(\lambda')\rangle_{L^2(\mathbb{R})} = \langle q_{IV}(\lambda), \psi_{IV}(\lambda')\rangle_{L^2(\mathbb{R})} = 0, \\ \langle q_{III}(\lambda), \psi_{IV}(\lambda')\rangle_{L^2(\mathbb{R})} = \langle q_{IV}(\lambda), \psi_{III}(\lambda')\rangle_{L^2(\mathbb{R})} = \dfrac{2\pi}{(e^{\phi} - e^{-\phi})}\delta(\lambda - \lambda'), \end{cases} \quad \lambda, \lambda' \in [\lambda_0, \mu_0]. \tag{2.35}$$

Proposition 2.8. *The set* $\{q_1, q_2\}$ *is orthogonal to* $\{\psi_1, \psi_2\}$ *in* L^2_{per} *:*

$$\begin{cases} \langle q_1, \psi_1\rangle_{L^2_{\rm per}} = \langle q_2, \psi_2\rangle_{L^2_{\rm per}} = 0, \\ \langle q_1, \psi_2\rangle_{L^2_{\rm per}} = -\langle q_2, \psi_1\rangle_{L^2_{\rm per}} = -\dfrac{1}{2}\dfrac{d}{d\phi}\|u\|^2_{L^2_{\rm per}}, \end{cases} \tag{2.36}$$

where $-\frac{1}{2}\frac{d}{d\phi}\|u\|^2_{L^2} = \frac{\pi k}{\sinh^2\phi}$*, and the inner products of* $\{q_I(0), q_{II}(0)\}$ *and* $\{\psi_I(0), \psi_{II}(0)\}$ *in* L^2_{per} *are*

$$\begin{cases} \langle q_I(0), \psi_I(0)\rangle_{L^2_{\rm per}} = \dfrac{2\pi}{k}\dfrac{\lambda_0^2}{\mu_0^2}, \\ \langle q_{II}(0), \psi_{II}(0)\rangle_{L^2_{\rm per}} = \dfrac{2\pi}{k}\dfrac{\mu_0^2}{\lambda_0^2}, \\ \langle q_I(0), \psi_{II}(0)\rangle_{L^2_{\rm per}} = \langle q_{II}(0), \psi_I(0)\rangle_{L^2_{\rm per}} = \dfrac{2\pi}{k}. \end{cases} \tag{2.37}$$

2.4 Completeness relation in L^2_{per}

The solutions (2.11), (2.12), (2.27), and (2.30) form a basis in L^2_{per} in the sense that every element $f(x) \in L^2_{per}$ can be uniquely represented as a superposition of elements in the basis:

$$f(x) = \sum_{n\in\mathbb{N}}(a_n\psi_I(kn;x) + b_n\psi_{II}(kn;x)) + c_0 + c_1\psi_1 + c_2\psi_2, \quad x \in \mathbb{T}, \tag{2.38}$$

where $\mathbb{T}$ represents the periodic domain of period $\frac{2\pi}{k}$. We have set $t = 0$ and omitted the time dependence of eigenfunctions in the above relation. When the time dependence is included, (2.38)

represents a general solution of the adjoint linearized BO equation (2.8).
The orthogonality relations (2.32), (2.36), and (2.37) allow us to compute the coefficients in the decomposition (2.38):

$$a_n = \frac{k}{2\pi}\langle q_{II}(kn), f\rangle_{L^2_{per}}, \quad b_n = \frac{k}{2\pi}\langle q_{I}(kn), f\rangle_{L^2_{per}}, \quad n \in \mathbb{N}, \tag{2.39}$$

and

$$c_1 = -\frac{1}{\pi k}\sinh^2\phi\langle q_2, f\rangle_{L^2_{per}}, \quad c_2 = \frac{1}{\pi k}\sinh^2\phi\langle q_1, f\rangle_{L^2_{per}}. \tag{2.40}$$

There are two representations of the coefficient c_0 since $\psi_I(0)$ and $\psi_{II}(0)$ are constant proportional to 1; we split c_0 symmetrically to get

$$c_0 = \frac{k}{4\pi}\Big(\langle q_{II}(0), f\rangle_{L^2_{per}}\psi_I(0) + \langle q_I(0), f\rangle_{L^2_{per}}\psi_{II}(0)\Big). \tag{2.41}$$

Substituting (2.39), (2.40), and (2.41) into our decomposition (2.38), we obtain

$$\begin{aligned} f(x) = \frac{k}{2\pi}\oint f(y)\Bigg(&\sum_{n\in\mathbb{N}}(q_{II}(kn;y)\psi_I(kn;x) + q_I(kn;y)\psi_{II}(kn;x)) \\ &+\frac{1}{2}\left[q_{II}(0;y)\psi_I(0;x) + q_I(0;y)\psi_{II}(0;x)\right] \\ &+\frac{2}{k^2}\sinh^2\phi\left[-q_2(y)\psi_1(x) + q_1(y)\psi_2(x)\right]\Bigg)\,dy. \end{aligned} \tag{2.42}$$

The identity (2.42) holds for all $f \in L^2_{per}$ if and only if the following completeness relation is satisfied.

Proposition 2.9. *The eigenfunctions* (2.11)*,* (2.12)*,* (2.27)*, and* (2.30) *of the adjoint linearized BO equation and the eigenfunctions* (2.19)*,* (2.20)*,* (2.28)*, and* (2.29) *of the linearized BO equation satisfy the following completeness relation:*

$$\begin{aligned} &\sum_{n\in\mathbb{N}}(q_I(kn;x)\psi_{II}(kn;y) + q_{II}(kn;x)\psi_I(kn;y)) + \frac{1}{2}(q_I(0;x)\psi_{II}(0;y) + q_{II}(0;x)\psi_I(0;y)) \\ &\quad + \frac{2}{k^2}\sinh^2\phi(q_1(x)\psi_2(y) - q_2(x)\psi_1(y)) = \frac{2\pi}{k}\delta(x-y), \quad x, y \in \mathbb{T}. \end{aligned} \tag{2.43}$$

2.5 Completeness Relation in $L^2(\mathbb{R})$

Similarly to the basis in L^2_{per}, the solutions (2.11), (2.12), (2.13), and (2.14) form a basis in $L^2(\mathbb{R})$ in the sense that every element $f \in L^2(\mathbb{R})$ can be uniquely represented as a superposition of elements in the basis:

$$f(x) = \int_0^\infty [a(\lambda)\psi_I(\lambda;x) + b(\lambda)\psi_{II}(\lambda;x)]\, d\lambda \\ + \int_{\lambda_0}^{\mu_0} [c(\lambda)\psi_{III}(\lambda;x) + d(\lambda)\psi_{IV}(\lambda;x)]\, d\lambda, \qquad x \in \mathbb{R}. \tag{2.44}$$

The orthogonality conditions (2.31), (2.33), (2.34), and (2.35) allow us to compute the coefficients in the decomposition (2.44). We get

$$a(\lambda) = \frac{1}{2\pi}\langle q_{II}(\lambda), f\rangle_{L^2(\mathbb{R})}, \quad b(\lambda) = \frac{1}{2\pi}\langle q_I(\lambda), f\rangle_{L^2(\mathbb{R})}, \quad \lambda \in (0,\infty), \tag{2.45}$$

and

$$c(\lambda) = \frac{(e^\phi - e^{-\phi})}{2\pi}\langle q_{IV}(\lambda), f\rangle_{L^2(\mathbb{R})}, \quad d(\lambda) = \frac{(e^\phi - e^{-\phi})}{2\pi}\langle q_{III}(\lambda), f\rangle_{L^2(\mathbb{R})}, \quad \lambda \in (\lambda_0, \mu_0). \tag{2.46}$$

Substituting the coefficients (2.45) and (2.46) into the decomposition (2.44) yields

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} f(y) \left(\int_0^\infty [q_{II}(\lambda;y)\psi_I(\lambda;x) + q_I(\lambda;y)\psi_{II}(\lambda;x)]\, d\lambda \right. \\ \left. + (e^\phi - e^{-\phi}) \int_{\lambda_0}^{\mu_0} [q_{IV}(\lambda;y)\psi_{III}(\lambda;x) + q_{III}(\lambda;y)\psi_{IV}(\lambda;x)]\, d\lambda \right) dy. \tag{2.47}$$

The identity (2.47) holds for all $f \in L^2(\mathbb{R})$ if and only if the following completeness relation is satisfied.

Proposition 2.10. *The eigenfunctions* (2.11), (2.12), (2.13)*, and* (2.14) *of the adjoint linearized BO equation and the eigenfunctions* (2.19), (2.20), (2.21)*, and* (2.22) *of the linearized BO equation satisfy the following completeness relation:*

$$\int_0^\infty [q_{II}(\lambda;x)\psi_I(\lambda;y) + q_I(\lambda;x)\psi_{II}(\lambda;y)]\, d\lambda \\ + (e^\phi - e^{-\phi}) \int_{\lambda_0}^{\mu_0} [q_{IV}(\lambda;x)\psi_{III}(\lambda;y) + q_{III}(\lambda;x)\psi_{IV}(\lambda;y)]\, d\lambda = 2\pi\delta(x-y), \quad x,y \in \mathbb{R}. \tag{2.48}$$

3 The nonlocal derivative nonlinear Schrödinger equation

We consider the periodic travelling wave solution of the NDNLS$^{\pm}$ equation and perform linearization at this solution. The construction of a solution to the linearized NDNLS$^{\pm}$ equation and a solution to the adjoint linearized NDNLS$^{\pm}$ using eigenfunctions of the Lax operator is shown. We obtain additional solutions to the linearized equation by exploiting symmetries of the NDNLS$^{\pm}$ equation. Finally, we conjecture a completeness relation for L^2_{per} space using functions obtained from solutions of the linearized equations, with coefficients given by orthogonality relations.

3.1 The periodic travelling wave solution of the NDNLS$^{\pm}$ equation

We consider the NDNLS$^{\pm}$ equations, which are written in dimensionless form (1.2). The NDNLS$^{\pm}$ equation (1.2) is a compatibility condition of the following system of Lax equations [3, 10, 16]:

$$\begin{cases} ip_x + \lambda p + uq^+ = 0, \\ q^+ - \mu q^- + \sigma \bar{u} p = 0, \\ ip_t + \lambda^2 p + \lambda u q^+ + i(uq^+_x - u_x q^+) = 0, \\ iq^{\pm}_t - 2i\lambda q^{\pm}_x + q^{\pm}_{xx} + \sigma q^{\pm}[(\pm i + H)(|u|^2)_x] = 0, \end{cases} \tag{3.1}$$

where $q^{\pm}$ are analytic functions in $\mathbb{C}^{\pm}$, λ is the spectral parameter, and $\bar{u}$ is the complex conjugate of u.
There are three symmetries of the NDNLS$^{\pm}$ equation relevant to this work; the translational symmetries in x and t and the rotational phase symmetry in the complex plane. Given a solution $u(x,t)$, the function $e^{i\theta}u(x - ct + \xi_0)$ is a solution for $\theta, c, \xi_0 \in \mathbb{R}$. Without loss of generality, we can set $\theta = \xi_0 = 0$ and consider the solution $u(\xi)$ with the travelling wave coordinate $\xi = x - ct$.

The NDNLS$^{\pm}$ equation (1.2) admits a periodic travelling wave solution (see Proposition 1 of [3])

$$u(x,t) = e^{\frac{1}{2}(\psi-\phi)} \frac{1 + e^{ik\xi - \psi}}{1 + e^{ik\xi - \phi}}, \quad \overline{u}(x,t) = e^{-\frac{1}{2}(\psi-\phi)} \frac{1 + e^{ik\xi + \psi}}{1 + e^{ik\xi + \phi}} \tag{3.2}$$

and

$$|u(x,t)|^2 = 1 - \frac{\sigma k \sinh\phi}{\cos k\xi + \cosh\phi}, \tag{3.3}$$

for parameters $k, \phi > 0$, $\psi \in \mathbb{R}$, and travelling wave coordinate $\xi = x - ct$ where $c \in \mathbb{R}$ is the wave speed. The parameters ϕ and ψ are uniquely determined by the following relations:

$$e^{2\phi} = \frac{\mu_0(\lambda_0 - \sigma)}{\lambda_0(\mu_0 - \sigma)}, \quad e^{\psi} = \frac{\lambda_0}{\mu_0} e^{\phi}, \quad \lambda_0 = -\frac{1}{2}(c + k), \quad \mu_0 = -\frac{1}{2}(c - k). \tag{3.4}$$

Note that $k = \mu_0 - \lambda_0$. The parameters $k > 0$ and $c \in \mathbb{R}$ are further restricted based on σ as follows:

- If $\sigma = 1$, then $k \in (0,1)$ and $c \in (-2+k, -k)$.
- If $\sigma = -1$, then $k \in (0,\infty)$ and $c \in (k+2, \infty)$ or $c \in (-\infty, -k)$.

To perform analysis of the periodic travelling wave (3.2) we use the Hirota bilinear form

$$u(x-ct) = \frac{g(x-ct)}{f(x-ct)}, \quad \overline{u}(x-ct) = \frac{g'(x-ct)}{f'(x-ct)}, \quad |u(x-ct)|^2 = 1 - i\sigma \frac{\partial}{\partial x} \log \frac{f(x-ct)}{f'(x-ct)} \tag{3.5}$$

where

$$\begin{cases} f(x-ct) = 1 + e^{ik\xi - \phi}, \\ f'(x-ct) = 1 + e^{ik\xi + \phi}, \end{cases} \qquad \begin{cases} g(x-ct) = \gamma(1 + e^{ik\xi - \psi}), \\ g'(x-ct) = \gamma^{-1}(1 + e^{ik\xi + \psi}), \end{cases} \qquad \gamma = e^{\frac{1}{2}(\psi - \phi)}. \tag{3.6}$$

Since f only has zeros in $\mathbb{C}^-$ and f' only has zeros in $\mathbb{C}^+$, u is analytic in $\mathbb{C}^+$ and $\overline{u}$ is analytic in $\mathbb{C}^-$. Additionally, using the relations (3.4) we note that $\gamma = \sqrt{\frac{\lambda_0}{\mu_0}}$.
The Lax spectrum for the periodic travelling wave (3.2) solving the NDNLS$^{\pm}$ equation was recently found in [3] (see Proposition 2). Similarly to the BO equation, the Lax operator for the NDNLS$^{\pm}$ equation (1.2) is self-adjoint in both $L^2(\mathbb{R})$ and L^2_{per}, so both spectra will be a subset of $\mathbb{R}$. For eigenfunctions of the Lax spectrum (3.1) in $L^2(\mathbb{R})$, the Lax spectrum of the periodic travelling wave (3.2) is

$$\sigma_L|_{L^2(\mathbb{R})} = [\lambda_0, \mu_0] \cup [\sigma, \infty), \tag{3.7}$$

for λ_0, μ_0 defined in (3.4). For eigenfunctions of the Lax system (3.1) in L^2_{per}, the Lax spectrum is given by

$$\sigma_L|_{L^2_{per}} = \{\lambda_0\} \cup \{\sigma + k(n-1)\}_{n\in\mathbb{N}}, \tag{3.8}$$

where $k = \mu_0 - \lambda_0$.

The linearized NDNLS$^{\pm}$ equation can be obtained by considering a perturbation q to the periodic travelling wave solution u (3.5) of the NDNLS$^{\pm}$ equation 1.2:

$$iq_t = q_{xx} + \sigma[u(i+H)(q\bar{u} + u\bar{q})_x + q(i+H)(|u|^2)_x]. \tag{3.9}$$

The adjoint linearized NDNLS$^{\pm}$ equation is not simply related to the linearized NDNLS$^{\pm}$ equation (3.9) like the derivative relationship between solutions of the adjoint linearized BO equation and solutions of the linearized BO equation in the previous section. In this work, the explicit form of the adjoint linearized NDNLS$^{\pm}$ equation has not yet been considered.

3.2 Eigenfunctions

When u is the travelling periodic wave (3.5), the Lax system (3.1) admits the following two sets of eigenfunctions [3]:

$$\begin{cases} p_I(\lambda, x, t) = e^{i(\lambda-\sigma)x+i(\lambda^2-1)t}\dfrac{f'(x-ct)}{f(x-ct)}, \\ q_I^+(\lambda; x, t) = -\sigma e^{i(\lambda-\sigma)x+i(\lambda^2-1)t}\dfrac{g'(x-ct)}{f(x-ct)}, \qquad \lambda \in [\sigma, \infty) \\ q_I^- = 0, \end{cases} \tag{3.10}$$

and

$$\begin{cases} p_{II}(\lambda; x, t) = -\dfrac{1}{\lambda f(x-ct)}\left(\gamma + \dfrac{\lambda-\lambda_0}{\lambda-\mu_0}(g(x-ct)-\gamma)\right), \\ q_{II}^+(\lambda; x, t) = \dfrac{1}{f(x-ct)}\left(1 + \dfrac{\lambda-\lambda_0}{\lambda-\mu_0}(f(x-ct)-1)\right), \qquad \lambda \in \mathbb{R}\setminus\{\mu_0\} \\ q_{II}^-(\lambda; x, t) = \dfrac{1}{f'(x-ct)}\left(1 + \dfrac{\lambda-\lambda_0}{\lambda-\mu_0}(f'(x-ct)-1)\right). \end{cases} \tag{3.11}$$

Based on the calculations in [11] for solitons, we have derived a solution to the linearized NDNLS$^\pm$ equation (3.9) constructed from eigenfunctions of the Lax system (3.10) and (3.11). The argument $x - ct$ is omitted in the proof for clarity.

Proposition 3.1. *The linearized NDNLS$^\pm$ equation admits the following solution for $\lambda \in [\sigma, \infty)$:*

$$w(\lambda; x, t) = \frac{f'(x-ct)}{f(x-ct)}e^{i(\lambda-\sigma)x+i(\lambda^2-1)t}\left(1 + \frac{k}{(\lambda-\mu_0)f(x-ct)}\right) = p_I\overline{q_{II}^-}. \tag{3.12}$$

Proof. First, we derive the form of $\overline{q_{II}^-}$ used in (3.12) from (3.11) using $\frac{\bar{f}'-1}{\bar{f}'} = \frac{1}{f}$ and $\frac{1}{\bar{f}'} = \frac{f-1}{f}$.

$$\begin{aligned} \overline{q_{II}^-} &= \frac{1}{\bar{f}'} + \frac{\lambda-\lambda_0}{\lambda-\mu_0}\cdot\frac{\bar{f}'-1}{\bar{f}'} \\ &= \frac{f-1}{f} + \frac{\lambda-\lambda_0}{(\lambda-\mu_0)f} \\ &= \frac{f(\lambda-\mu_0)-(\lambda-\mu_0)+\lambda-\lambda_0}{(\lambda-\mu_0)f} \\ &= 1 + \frac{k}{(\lambda-\mu_0)f}. \end{aligned}$$

Thus, $w = p_I\overline{q_{II}^-}$, and w is analytic in $\mathbb{C}^+$ since both p_I and $\overline{q_{II}^-}$ are analytic in $\mathbb{C}^+$. To simplify notation, in the remainder of the proof we neglect the subscripts and refer to p_I as p and $\overline{q_{II}^-}$ as $\overline{q^-}$.

To show w solves (3.9), we use the Lax system (3.1) and the expression $p_x = p(i(\lambda-\sigma)+\frac{f'_x}{f'}-\frac{f_x}{f})$. Substituting w into the linearized NDNLS$^{\pm}$ equation (3.9), we obtain:

$$i(p_t\overline{q^-}+p\overline{q_t^-}) = p_{xx}\overline{q^-}+2p_x\overline{q_x^-}+p\overline{q_{xx}^-}+\sigma[u(i+H)(\bar{u}p\overline{q^-}+u\bar{p}q^-)_x+p\overline{q^-}(i+H)(|u|^2)_x].$$

To extract terms from the Hilbert transform, we determine analyticity in $\mathbb{C}^{\pm}$ and use the definition for analytic functions in $\mathbb{C}^{\pm}$. The eigenfunction q^- is analytic in $\mathbb{C}^-$ and the factor $\frac{1}{f}$ in u is cancelled by the complex conjugate of $\frac{f'}{f}$ in $\overline{p}$, leaving $(f')^2$ in the denominator of $u\overline{p}q^-$. Hence $u\overline{p}q^-$ is analytic in $\mathbb{C}^-$ and $(i+H)(u\overline{p}q^-)_x = 0$. For the other term $\overline{u}p\overline{q^-}$ in the Hilbert transform, the factor $\frac{1}{f'}$ in $\overline{u}$ is cancelled by $\frac{f'}{f}$ in p, leaving f^2 in the denominator. Thus, $\overline{u}p\overline{q^-}$ is analytic in $\mathbb{C}^+$ and $(i+H)(\overline{u}p\overline{q^-})_x = 2i(\overline{u}p\overline{q^-})_x$.

Using the complex conjugate of the fourth equation of the Lax system (3.1), we obtain

$$-ip\overline{q_t^-}+p\overline{q_{xx}^-}+\sigma p\overline{q^-}(i+H)(|u|^2)_x) = p(-2i\lambda\overline{q_x^-}).$$

After expressing the derivative in x of $\overline{u}p\overline{q^-}$, the above simplifications leave us with

$$ip_t\overline{q^-} = p_{xx}\overline{q^-}+2p_x\overline{q_x^-}+2i\sigma u(\bar{u}_xp\overline{q^-}+\bar{u}p_x\overline{q^-}+\bar{u}p\overline{q_x^-})-2i\lambda p\overline{q_x^-}.$$

Combining terms with factors of $\overline{q_x^-}$ and using the definition (3.5) for $|u|^2$ and the expression for p_x, we have

$$\overline{q_x^-}(2p_x+2i\sigma u\bar{u}p-2i\lambda p) = 2\overline{q_x^-}p\left[i(\lambda-\sigma)+\frac{f'_x}{f'}-\frac{f_x}{f}+i\sigma\left(1-i\sigma\left(\frac{f_x}{f}-\frac{f'_x}{f'}\right)\right)-i\lambda\right] = 0.$$

All remaining terms have a factor of $\overline{q^-}$; we cancel to obtain $ip_t = p_{xx}+2i\sigma u(p\bar{u}_x+p_x\bar{u})$. We differentiate the first equation and use the third equation in the Lax system (3.1), which yields the following:

$$p_{xx} = i(\lambda p_x+u_xq^++uq_x^+), \quad ip_t = -\lambda^2p-\lambda uq^+-i(uq_x^+-u_xq^+),$$

where q^+ corresponds to q_I^+. Substituting these expressions into our equation, we have

$$0 = \lambda^2p+\lambda uq^++2iuq_x^++i\lambda p_x+2i\sigma u(p\overline{u}_x+p_x\overline{u}).$$

Using the first equation and differentiating the second equation in the Lax system (3.1), we have

$$uq^+ = -ip_x-\lambda p, \quad q_x^+ = -\sigma(\overline{u}_xp+p_x\overline{u}),$$

where we have dropped the μq^- term in the second equation since $q_I^- = 0$. Substituting into our equation, we obtain

$$\lambda^2p+\lambda(-ip_x-\lambda p)+2iu(-\sigma(\overline{u}_xp+p_x\overline{u}))+i\lambda p_x+2i\sigma u(\overline{u}_xp+p_x\overline{u}) = 0.$$

□

Remark 3.2. *The solution* $w = p_I \overline{q_{II}^-}$ (3.12) *to the linearized NDNLS*$^\pm$ *equation is identical to the solution* $q_I \cong \varphi_I^+ \overline{\varphi_{II}^-}$ (2.19) *to the linearized BO equation with the exception of* λ *in the exponent shifted to* $\lambda - \sigma$ *and* λ^2 *shifted to* $(\lambda - \sigma)(\lambda + \sigma) = \lambda^2 - 1$.

Again following the procedure in [11], we conjecture that the following function satisfies the adjoint linearized NDNLS$^\pm$ equation. Note that we have not yet derived the adjoint linearized NDNLS$^\pm$ equation; this conjecture is based on [11] and our previous work on the BO equation (2.12).

Conjecture 3.3. *The adjoint linearized NDNLS*$^\pm$ *equation admits the following solution for* $\lambda \in [\sigma, \infty)$:

$$\tilde{w}(\lambda; x, t) = e^{-i(\lambda-\sigma)x - i(\lambda^2-1)t} \frac{f(x-ct)}{f'(x-ct)} \left(1 - \frac{k}{(\lambda - \lambda_0) f(x-ct)}\right) = \overline{p_I} q_{II}^+, \tag{3.13}$$

where constants have been omitted from q_{II}^+ *and* $\overline{p_I}$.

Remark 3.4. *The proposed solution* $\tilde{w} = \overline{p_I} q_{II}^+$ (3.13) *to the adjoint linearized NDNLS*$^\pm$ *equation is identical to the solution* $\psi_{II} \cong \overline{\varphi_I^+} \varphi_{II}^+$ (2.12) *of the adjoint linearized BO equation with the exception of the same shifts of* λ *in the exponent as in Remark 3.2.*

For L^2_{per}, following [11] and our work for the BO equation, we can find eigenfunctions corresponding to the discrete spectrum using symmetries of the NDNLS$^\pm$ equation (1.2) discussed in section 3.1. Taking derivatives of solutions to a partial differential equation with respect to the symmetry parameters yields solutions to the linearized equation. Therefore, if we consider the travelling periodic wave in the form $e^{i\theta} u(x - ct + \xi_0)$, we have the following solutions to the linearized NDNLS$^\pm$ equation (3.9):

$$\partial_{\xi_0}(e^{i\theta} u(x - ct + \xi_0)), \quad \partial_c(e^{i\theta} u(x - ct + \xi_0)), \quad \text{and} \quad \partial_\theta(e^{i\theta} u(x - ct + \xi_0))$$

We can observe that $\partial_{\xi_0}(e^{i\theta} u(x - ct + \xi_0)) = \partial_x(e^{i\theta} u(x - ct + \xi_0))$. The representation of the periodic travelling wave in (3.5) can be recovered for these derivative solutions by setting $\xi_0 = \theta = 0$ and setting c to the desired wave speed after taking the derivative. It is important to note that taking the derivative of the periodic travelling wave with respect to the fourth parameter k does not produce a valid solution of the linearized NDNLS$^\pm$ equation (3.9) because it is not periodic. We explicitly compute the derivatives in order to represent them in terms of eigenfunctions of the Lax equation. Using $\xi = x - ct$, we have for $\partial_x(e^{i\theta} u(\xi + \xi_0))$:

$$\begin{aligned} \partial_x(e^{i\theta} u(\xi + \xi_0)) &= \partial_x e^{i\theta} \left(\frac{g(\xi + \xi_0)}{f(\xi + \xi_0)}\right) \\ &= e^{i\theta} \frac{f g_x - g f_x}{f^2} \end{aligned}$$

$$= e^{i\theta}\frac{ik\gamma}{f^2}(e^{ik(\xi+\xi_0)-\psi} + e^{2ik(\xi+\xi_0)-\psi-\phi} - e^{ik(\xi+\xi_0)-\phi} - e^{2ik(\xi+\xi_0)-\psi-\phi})$$
$$= e^{i\theta}\frac{ik}{f^2}(\gamma e^{ik(\xi+\xi_0)-\psi} + \gamma - \gamma - \gamma e^{ik(\xi+\xi_0)-\phi})$$
$$= e^{i\theta}\frac{ik}{f(\xi+\xi_0)}(u(\xi+\xi_0)-\gamma)$$
$$= \frac{ik}{f(\xi)}(u(\xi)-\gamma) = \partial_x u(\xi),$$

where the last line is obtained by setting $\xi_0 = \theta = 0$. The derivative with respect to θ evaluated at $\xi_0 = \theta = 0$ is

$$\partial_\theta(e^{i\theta}u(\xi+\xi_0))|_{\theta=0,\ \xi_0=0} = ie^{i\theta}u(\xi+\xi_0)|_{\theta=0,\ \xi_0=0} = iu(\xi).$$

To take the derivative of u with respect to c, we must use the chain rule as ξ, ϕ, and ψ are functions of c:

$$\begin{aligned}
\partial_c(e^{i\theta}u(\xi+\xi_0)) &= \partial_c\left(e^{i\theta}e^{\frac{1}{2}(\psi-\phi)}\frac{1+e^{ik(\xi+\xi_0)-\psi}}{1+e^{ik(\xi+\xi_0)-\phi}}\right)\\
&= e^{i\theta}\left(\frac{\partial u(\xi+\xi_0)}{\partial\xi}\frac{\partial\xi}{\partial c} + \frac{\partial u(\xi+\xi_0)}{\partial\phi}\frac{\partial\phi}{\partial c} + \frac{\partial u(\xi+\xi_0)}{\partial\psi}\frac{\partial\psi}{\partial c}\right)\\
&= e^{i\theta}\left[-t\frac{\partial u(\xi+\xi_0)}{\partial\xi} + \frac{\partial\phi}{\partial c}\left(e^{\frac{1}{2}(\psi-\phi)}\left(-\frac{1}{2}\cdot\frac{1+e^{ik(\xi+\xi_0)-\psi}}{1+e^{ik(\xi+\xi_0)-\phi}} + e^{ik(\xi+\xi_0)-\phi}\frac{1+e^{ik(\xi+\xi_0)-\psi}}{(1+e^{ik(\xi+\xi_0)-\phi})^2}\right)\right)\right.\\
&\qquad\left.+\frac{\partial\psi}{\partial c}\left(e^{\frac{1}{2}(\psi-\phi)}\left(\frac{1}{2}\cdot\frac{1+e^{ik(\xi+\xi_0)-\psi}}{1+e^{ik(\xi+\xi_0)-\phi}} - \frac{e^{ik(\xi+\xi_0)-\psi}}{1+e^{ik(\xi+\xi_0)-\phi}}\right)\right)\right]\\
&= e^{i\theta}\left[-t\frac{\partial u(\xi+\xi_0)}{\partial\xi} + \frac{\partial\phi}{\partial c}\left(-\frac{1}{2}u(\xi+\xi_0) + u(\xi+\xi_0)\frac{f(\xi+\xi_0)-1}{f(\xi+\xi_0)}\right)\right.\\
&\qquad\left.+\frac{\partial\psi}{\partial c}\left(\frac{1}{2}u(\xi+\xi_0) - \frac{g(\xi+\xi_0)-\gamma}{f(\xi+\xi_0)}\right)\right]\\
&= -t\frac{\partial u(\xi)}{\partial\xi} + \frac{\partial\phi}{\partial c}\left(\frac{u(\xi)}{2} - \frac{u(\xi)}{f(\xi)}\right) + \frac{\partial\psi}{\partial c}\left(\frac{\gamma}{f(\xi)} - \frac{u(\xi)}{2}\right),
\end{aligned}$$

where again the last line is obtained by setting $\xi_0 = \theta = 0$.
Using the relations in (3.4), we can represent ψ and ϕ in terms of c as

$$\phi = \ln\left[\left(\frac{(c-k)(c+k+2\sigma)}{(c+k)(c-k+2\sigma)}\right)^{\frac{1}{2}}\right], \quad \text{and} \quad \psi = \ln\left[\left(\frac{(c+k)(c+k+2\sigma)}{(c-k)(c-k+2\sigma)}\right)^{\frac{1}{2}}\right],$$

and so

$$\frac{\partial\phi}{\partial c} = \frac{k}{c^2-k^2} - \frac{k}{(c+2\sigma)^2-k^2}, \quad \frac{\partial\psi}{\partial c} = -\frac{k}{c^2-k^2} - \frac{k}{(c+2\sigma)^2-k^2}.$$

Since $\frac{\partial u}{\partial\xi} = \frac{\partial u}{\partial x}$, we finally have

$$\partial_c(e^{i\theta}u(\xi+\xi_0))|_{\theta=0,\ \xi_0=0} = -t\frac{\partial u}{\partial x} - \frac{u}{f}\frac{\partial\phi}{\partial c} + \frac{u}{2}\left(\frac{\partial\phi}{\partial c} - \frac{\partial\psi}{\partial c}\right) + \frac{\gamma}{f}\frac{\partial\psi}{\partial c}.$$

Note that the first term in the above equation is linearly dependent with $\partial_x u$; we can omit this term from the function used in the completeness relation. We have the following result:

Proposition 3.5. *The following functions related to solutions of the linearized NDNLS$^{\pm}$ equation* (3.9) *will be used in a completeness relation for L^2_{per} :*

$$
\begin{aligned}
h_1 &= \frac{ik}{f}(u-\gamma) \\
h_2 &= iu \\
h_3 &= -\frac{u}{f}\frac{\partial \phi}{\partial c} + \frac{u}{2}\left(\frac{\partial \phi}{\partial c} - \frac{\partial \psi}{\partial c}\right) + \frac{\gamma}{f}\frac{\partial \psi}{\partial c}.
\end{aligned}
\tag{3.14}
$$

Remark 3.6. *The functions in Proposition 3.5* (3.14) *can be represented in terms of eigenfunctions of the Lax system* (3.1)*:*

$$
\begin{aligned}
h_1 &= -i \lim_{\lambda \to \mu_0} [(\lambda-\mu_0)^2 p_{II}\overline{q_{II}^-}] \\
h_2 &= i(-\sigma)e^{-\psi-\phi}\, p_I \overline{q_I^+} \\
h_3 &= \frac{1}{k}\frac{\partial \phi}{\partial c}\lim_{\lambda \to \mu_0}[(\lambda-\mu_0)^2 p_{II}\overline{q_{II}^-}] + \frac{\gamma}{k}\left(\frac{\partial \psi}{\partial c} - \frac{\partial \phi}{\partial c}\right)\lim_{\lambda \to \mu_0}[(\lambda-\mu_0)\overline{q_{II}^-}] \\
&\quad + \frac{1}{2}\left(\frac{\partial \psi}{\partial c} - \frac{\partial \phi}{\partial c}\right)(-\sigma)e^{-\psi-\phi}\, p_I\overline{q_I^+}
\end{aligned}
\tag{3.15}
$$

It is interesting to note that h_2 only uses eigenfunctions valid for $\lambda \in [0,\infty)$.

Remark 3.7. *Unlike analysis performed for the BO equation, The imaginary unit i cannot be treated as a constant for the NDNLS$^{\pm}$ equation due to complex conjugation. Thus h_1 and h_2 cannot be fully represented by eigenfunctions of the Lax system in this sense.*

3.3 Orthogonality of Eigenfunctions

The orthogonality of functions in $L^2(\mathbb{R})$ and L^2_{per} is defined in the same way as the orthogonality of eigenfunctions of the BO equation in section 2.3. We have the following orthogonality result for the eigenfunctions w (3.12) and $\tilde{w}$ (3.13) which follows from the orthogonality of q_I and ψ_{II} in the BO equation:

Proposition 3.8. *The solution w of the linearized NDNLS equation* (3.9) *and the proposed solution $\tilde{w}$ of the adjoint linearized NDNLS equation are orthogonal to each other in $L^2(\mathbb{R})$ and L^2_{per} :*

$$
\langle w(\lambda), \tilde{w}(\lambda')\rangle_{L^2(\mathbb{R})} = 2\pi\delta(\lambda-\lambda'), \quad \lambda,\lambda' \in [\sigma,\infty), \tag{3.16}
$$

$$
\langle w(kn), \tilde{w}(km)\rangle_{L^2_{per}} = \frac{2\pi}{k}\delta_{nm}, \quad n,m \in \mathbb{Z},\ n,m \geq \sigma \tag{3.17}
$$

where $\delta(\lambda-\lambda')$ is the Dirac delta distribution and δ_{nm} is the Kronecker delta.

Proof. To prove (3.16), we can directly compute the integral in the sense of distributions. We will denote $e^{i(\lambda-\lambda')x+i(\lambda^2-(\lambda')^2)t}$ by $\mathbf{E}$.

$$\begin{aligned}\langle w(\lambda),\tilde{w}(\lambda')\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \mathbf{E}\,\frac{f'}{f}\left(1+\frac{k}{(\lambda-\mu_0)f}\right)\frac{f}{f'}\left(1-\frac{k}{(\lambda'-\lambda_0)f}\right)dx\\ &= \int_{\mathbb{R}} \mathbf{E}\left(1+\frac{k}{(\lambda-\mu_0)f}-\frac{k}{(\lambda'-\lambda_0)f}-\frac{k^2}{(\lambda'-\lambda_0)(\lambda-\mu_0)f^2}\right)dx.\end{aligned}$$

From the first term, we get $\int_{\mathbb{R}} e^{i(\lambda-\lambda')x+i(\lambda^2-(\lambda')^2)t}dx = 2\pi\delta(\lambda-\lambda')$. We simplify the last three terms of the integral to get

$$\begin{aligned}&\int_{\mathbb{R}} \mathbf{E}\left(\frac{k}{(\lambda-\mu_0)f}-\frac{k}{(\lambda'-\lambda_0)f}-\frac{k^2}{(\lambda'-\lambda_0)(\lambda-\mu_0)f^2}\right)dx\\ &= \frac{ik}{(\lambda'-\lambda_0)(\lambda-\mu_0)}\int_{\mathbb{R}} \mathbf{E}\left(\frac{i(\lambda-\lambda')}{f}-\frac{ik}{f}+\frac{ik}{f^2}\right)dx\\ &= \frac{ik}{(\lambda'-\lambda_0)(\lambda-\mu_0)}\int_{\mathbb{R}} \mathbf{E}\left(\frac{i(\lambda-\lambda')}{f}-\frac{f_x}{f^2}\right)dx\\ &= \frac{ik}{(\lambda'-\lambda_0)(\lambda-\mu_0)}\int_{\mathbb{R}} \frac{\partial}{\partial x}\left(\frac{\mathbf{E}}{f}\right)dx\\ &= 0,\end{aligned}$$

since $\mathbf{E}|_{x\to-\infty}^{x\to\infty} = 0$ in the sense of distributions. The computation for (3.17) is the same after interchanging λ, λ' with kn, km, and the integration performed in the periodic domain gives

$$\oint e^{ik(n-m)x+ik^2(n^2-m^2)t}dx = \frac{2\pi}{k}\delta_{nm}, \quad \text{and} \quad \oint \frac{\partial}{\partial x}\left(\frac{e^{ik(n-m)x+ik^2(n^2-m^2)t}}{f}\right)dx = 0.$$

□

4 Future Work

4.1 Completeness in L^2_{per}

To establish a completeness relation in L^2_{per}, it remains to

1. Find the remaining adjoint eigenfunctions corresponding to the discrete spectrum,
2. Show the proposed solution $\tilde{w}$ and the adjoint eigenfunctions for the discrete spectrum satisfy the adjoint linearized NDNLS$^{\pm}$ equation,
3. Find orthogonality relations between adjoint and linearized eigenfunctions for the discrete spectrum, and
4. Prove a completeness relation with the appropriate eigenfunctions and constants.

There should be three eigenfunctions $\tilde{h}_1, \tilde{h}_2$, and $\tilde{h}_3$ of the adjoint linearized NDNLS$^{\pm}$ equation along with the three eigenfunctions h_1, h_2, and h_3 of the linearized NDNLS$^{\pm}$ equation. Orthogonality relations between the sets $\{h_1, h_2, h_2\}$ and $\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\}$ will form a 3×3 projection matrix, from which we can deduce coefficients c_1, c_2, and c_3 for use in the completeness relation. This should give us a completeness relationship of the following form:

Conjecture 4.1. *The solution* (3.12) *to the linearized NDNLS*$^{\pm}$ *equation and the functions in* (3.14) *form a completeness relation in the sense that every element* $f(x) \in L^2_{per}$ *can be uniquely represented as a superposition of these functions:*

$$f(x) = \sum_{n=\sigma}^{\infty} a_n w(kn; x, t) + c_1 h_1 + c_2 h_2 + c_3 h_3, \quad x \in \mathbb{T}, \tag{4.1}$$

where the coefficient a_n *is given by* $\frac{k}{2\pi}\langle \tilde{w}(kn), f\rangle$ *and the coefficients* c_1, c_2, *and* c_3 *will be determined from the orthogonality relations between* h_1, h_2, h_3, *and* $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$.

Remark 4.2. *In [11], the NDNLS*$^{\pm}$ *equation was considered for soliton solutions, and a completeness relation was shown. However, we were unable to apply all the ideas in this work to form a completeness relation in* L^2_{per} *space. There are only two eigenfunctions corresponding to the discrete spectrum in [11] despite there being three symmetries and thus three corresponding solutions of the linearized NDNLS*$^{\pm}$ *equation. Furthermore, we were unable to understand the orthogonality relations between eigenfunctions in the discrete spectrum (equations* $20,\ 21$ *and* $23,\ 24$ *in [11]) as it appears that some orthogonality relations are nonzero despite being required to be zero by the completeness relation.*

4.2 Completeness in $L^2(\mathbb{R})$

The steps remaining to prove a completeness relation in $L^2(\mathbb{R})$ are very similar to the work remaining for L^2_{per} space, with the addition of needing to find solutions to the linearized NDNLS$^\pm$ equation corresponding to $\lambda \in (\lambda_0, \mu_0)$. Since the solutions of the linearized and adjoint linearized NDNLS$^\pm$ and BO equations were identical for $\lambda \in [\sigma, \infty)$ and $\lambda \in [0, \infty)$, it is reasonable to expect that the solutions of the linearized and adjoint linearized NDNLS$^\pm$ equations for $\lambda \in (\lambda_0, \mu_0)$ may be given by the same formulas as one of the solutions q_{III} or q_{IV} (equations (2.25) and (2.26)) for the linearized equation, and one of the solutions ψ_{III} or ψ_{IV} (equations (2.17) and (2.18)) for the adjoint linearized equation. We can identify corresponding eigenfunctions of the Lax operators for each equation by $\varphi_I^+ \leftrightarrow p_I$, $\varphi_{II}^+ \leftrightarrow q_{II}^+$, and $\varphi_{II}^- \leftrightarrow q_{II}^-$.
Once these functions have been found, it will remain to determine the orthogonality relations and define a completeness relation similar in form to (2.48) using the orthogonality relations to determine the appropriate coefficients.

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