Algebraic solitons in the massive Thirring model

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April 22, 2025

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The Massive Thirring Model

This work is devoted to the algebraic solitons in the massive Thirring model (MTM) written in laboratory coordinates:

$$\begin{cases} \mathrm{i}(u_t + u_x) + v = |v|^2 u \\ \mathrm{i}(v_t - v_x) + u = |u|^2 v \end{cases} \tag{1}$$

where $(u, v) \in \mathbb{C}^2$ and subscripts denote partial derivatives in $(x, t) \in \mathbb{R}^2$.
We would like to construct rational solutions to the MTM system (1) and to understand dynamics of algebraic solitons.

Motivation

Similar rational solutions of the MTM system for rogue waves were studied:

- J. Chen, B. Yang, and B.-F. Feng, Rogue waves in the massive Thirring model, Stud Appl Math. 151, 1020 (2023).
- L. Guo, L. Wang, Y. Cheng, and J. He, High-order rogue wave solutions of the classical massive Thirring model equations, Commun. Nonlinear Sci. Numer. Simul. 52, 11 (2017).
- Y. L. Ye, L. L. Bu, C. C. Pan, S. H. Chen, D. Mihalache, and F. Baronio, Super rogue wave states in the classical massive Thirring model system, Rom. Rep. Phys. 73, 117 (2021).

Compared to these solutions, our rational solution for the algebraic double soliton is not a rogue wave since it describes two algebraic solitons at the trivial, modulationally stable background.

The modified Korteweg–de Vries (mKdV) equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{2}$$

has the algebraic soliton and the second-order rational solution (by Chowdury, Ankiewicz & Akhmediev(2016)):

$$u_1(x,t) = 1 - \frac{4}{1+4(x-6t)^2}, u_2(x,t) = 1 + 12\frac{G}{D},$$

where

$$G = 3 - 8(-x+t)[2(-x+t)^3 + 3(-\frac{11}{3}x + t)],$$

$$D = -8x[48x^4(-\frac{1}{6}x + t) - 2x^3(60t^2 - 13) + 8x^2 t(20t^2 - 9) - \frac{1}{2}x(240t^4 - 120t^2 + 139) \\ + t(48t^4 - 8t^2 + 51)] + 64t^6 + 48t^4 + 108t^2 + 9.$$

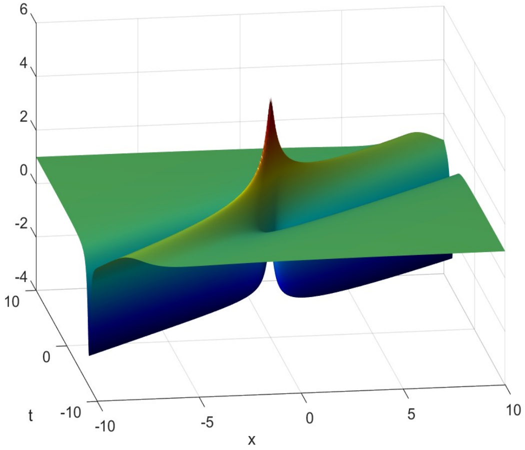

Figure 1: The solution surface for the second order mKdV rational solution.

Stability of Algebraic Solitons in MTM

Every method of nonlinear analysis known in the theory of integrable systems fails to prove the stability of algebraic solitons in MTM:

- Coercivity of the energy function required for the proof of Lyapunov stability holds for exponential solitons (Pelinovsky and Shimabukuro (2014)) but fails for algebraic solitons.
- The Darboux transformation proves the stability of exponential solitons in the MTM system (Contreras, Pelinovsky and Shimabukuro (2016)), but does not generate algebraic solitons.
- Algebraic solitons decay so slowly that they are not satisfied with developing the inverse scattering transform (IST).

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Main Result

A normalized family of exponential solitons of the MTM system (1) is given by the standing wave solutions of the form

$$\begin{bmatrix} u_{\mathrm{sol}}(x,t) \\ v_{\mathrm{sol}}(x,t) \end{bmatrix} = \sin\gamma \begin{bmatrix} \mathrm{sech}\left(x\ \sin\gamma + \frac{\mathrm{i}\gamma}{2}\right) \\ \mathrm{sech}\left(x\ \sin\gamma - \frac{\mathrm{i}\gamma}{2}\right) \end{bmatrix} e^{\mathrm{i}t\cos\gamma}, \quad \gamma \in (0,\pi). \tag{3}$$

The parameter γ defines the frequency parameter $\omega := \cos(\gamma)$ of the exponential solitons, which is chosen in the gap $(-1,1)$ of the frequency spectrum of the linear Dirac operator

$$\mathcal{D} := \begin{bmatrix} \mathrm{i}\partial_x & 1 \\ 1 & -\mathrm{i}\partial_x \end{bmatrix}.$$

The family (3) reduces to the small-amplitude, long-scale, sech-shaped soliton with $\sigma = +1$ as $\omega \to 1$ ($\gamma \to 0$) and to the finite-amplitude, finite-scale, algebraic soliton

$$\gamma = \pi : \quad \begin{bmatrix} u_{\mathrm{alg}}(x,t) \\ v_{\mathrm{alg}}(x,t) \end{bmatrix} = \begin{bmatrix} \dfrac{2}{1+2\mathrm{i}x} \\ \dfrac{2}{1-2\mathrm{i}x} \end{bmatrix} e^{-\mathrm{i}t} \tag{4}$$

as $\omega \to -1$.

To construct the second-order rational solution, we use the bilinear formulation of MTM system (1) given by Chen & Feng (2023).

The MTM system (1) is transformed to a system of bilinear equations by the following transformation,

$$u = \frac{g}{\bar{f}}, \qquad v = \frac{h}{f}, \tag{5}$$

Substituting (5) into (1) yields the following system of bilinear equations for f, h, and g:

$$\left.\begin{aligned} \mathrm{i}f(g_t + g_x) - \mathrm{i}g(f_t + f_x) + h\bar{f} = 0, \\ \mathrm{i}\bar{f}(h_t - h_x) - \mathrm{i}h(\bar{f}_t - \bar{f}_x) + gf = 0, \\ \mathrm{i}\bar{f}(f_x + f_t) - \mathrm{i}f(\bar{f}_t + \bar{f}_x) - |h|^2 = 0, \\ \mathrm{i}f(\bar{f}_t - \bar{f}_x) - \mathrm{i}\bar{f}(f_t - f_x) - |g|^2 = 0. \end{aligned}\right\} \tag{6}$$

To obtain the new solutions of the MTM system (1), we gonna do the following steps:

- Using a new parameterization of the exponential two-soliton solutions with 8 parameters.
- Taking the limits to the algebraic two-soliton solutions.

The exact formula for the algebraic two-soliton solution of the MTM system (1) can be written in the form by taking the limit $\gamma_{1,2} \to \pi$:

$$u(x,t) = -\frac{2\mathrm{i}\delta_1^{-1/2} e^{-\mathrm{i}T_1}\left[2X_2 + \mathrm{i} - \frac{4\mathrm{i}\delta_1}{\delta_1-\delta_2}\right] + 2\mathrm{i}\delta_2^{-1/2} e^{-\mathrm{i}T_2}\left[2X_1 + \mathrm{i} + \frac{4\mathrm{i}\delta_2}{\delta_1-\delta_2}\right]}{(2X_1 - \mathrm{i})(2X_2 - \mathrm{i}) + \frac{4\sqrt{\delta_1\delta_2}}{(\delta_1-\delta_2)^2}\left[\sqrt{\delta_1}e^{\frac{\mathrm{i}}{2}(T_1-T_2)} - \sqrt{\delta_2}e^{-\frac{\mathrm{i}}{2}(T_1-T_2)}\right]^2}, \tag{7}$$

$$v(x,t) = \frac{2\mathrm{i}\delta_1^{1/2} e^{-\mathrm{i}T_1}\left[2X_2 - \mathrm{i} - \frac{4\mathrm{i}\delta_2}{\delta_1-\delta_2}\right] + 2\mathrm{i}\delta_2^{1/2} e^{-\mathrm{i}T_2}\left[2X_1 - \mathrm{i} + \frac{4\mathrm{i}\delta_1}{\delta_1-\delta_2}\right]}{(2X_1 + \mathrm{i})(2X_2 + \mathrm{i}) + \frac{4\sqrt{\delta_1\delta_2}}{(\delta_1-\delta_2)^2}\left[\sqrt{\delta_1}e^{-\frac{\mathrm{i}}{2}(T_1-T_2)} - \sqrt{\delta_2}e^{\frac{\mathrm{i}}{2}(T_1-T_2)}\right]^2} \tag{8}$$

$$\text{where } X_j = \frac{x + c_j t}{\sqrt{1-c_j^2}} + x_j, \quad T_j = \frac{t + c_j x}{\sqrt{1-c_j^2}} + t_j, \quad c_j = \frac{\delta_j^2 - 1}{\delta_j^2 + 1}$$

with $\delta_{1,2} > 0$ such that $\delta_1 \neq \delta_2$ and translational parameters $x_{1,2} \in \mathbb{R}$ and $t_{1,2} \in \mathbb{R}$.

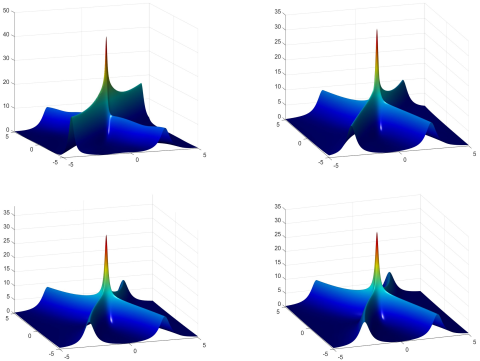


Figure 2: The solution surface for $|u|^2 + |v|^2$ versus (x, t) for the family (7) and (8) with $x_1 = x_2 = t_1 = t_2 = 0$ and $\delta_1 = 1 + \varepsilon$, $\delta_2 = 1 - \varepsilon$ with $\varepsilon = 0.75$ (top left), $\varepsilon = 0.5$ (top right), $\varepsilon = 0.25$ (bottom left), and $\varepsilon = 0.01$ (bottom right).

In the limit $\delta_1, \delta_2 \to 1$ of the algebraic two-soliton solution given by (7) and (8), we derived the following new rational solution to the MTM system (1):

$$\begin{bmatrix} u_{\rm double}(x,t) \\ v_{\rm double}(x,t) \end{bmatrix} = \begin{bmatrix} \dfrac{4(-3+6\mathrm{i}x-12x^2-8\mathrm{i}x^3-12t(2x-\mathrm{i})-\mathrm{i}\beta)}{3+24\mathrm{i}x-24x^2+32\mathrm{i}x^3-16x^4+48t^2+2\beta(2x-\mathrm{i})} \\ \dfrac{4(-3-6\mathrm{i}x-12x^2+8\mathrm{i}x^3+12t(2x+\mathrm{i})+\mathrm{i}\beta)}{3-24\mathrm{i}x-24x^2-32\mathrm{i}x^3-16x^4+48t^2+2\beta(2x+\mathrm{i})} \end{bmatrix} e^{-\mathrm{i}t}, \quad (9)$$

where $\beta \in \mathbb{R}$ is a free parameter of the family.
Taking $\delta_{1,2} = 1$ gives the algebraic double soliton with zero wave speed.

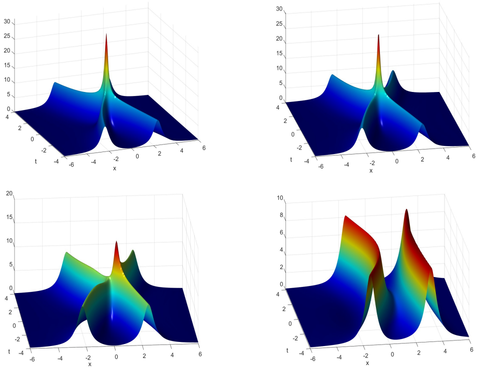


Figure 3: The solution surface for $|u|^2 + |v|^2$ versus (x, t) for the family (9) with $\beta = 0$ (top left), $\beta = 1$ (top right), $\beta = 10$ (bottom left), and $\beta = 100$ (bottom right).

Mass of the algebraic double-soliton

The mass for the MTM system (1) is defined by

$$Q(u,v) := \int_{\mathbb{R}} (|u|^2 + |v|^2)dx. \tag{10}$$

It follows from (3) that

$$|u(x,t)|^2 + |v(x,t)|^2 = \frac{4\sin^2\gamma}{\cos\gamma + \cosh(2x\ \sin\gamma)},$$

which implies that $Q_{\rm sol}(\gamma) := Q(u_{\rm sol}, v_{\rm sol}) = 4\gamma$ with the largest mass at $Q_{\rm sol}(\pi) = 4\pi$.

It follows from the bilinear equations that

$$|u|^2 + |v|^2 = \frac{|g|^2 + |h|^2}{|f|^2} = 2\mathrm{i}\left(\frac{f_x}{f} - \frac{\bar{f}_x}{\bar{f}}\right),$$

where

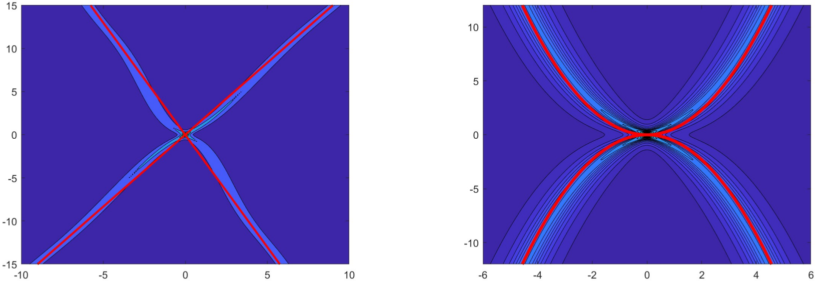
$$f = 16x^4 + 32\mathrm{i}x^3 + 24x^2 + 24\mathrm{i}x - 3 - 48t^2 - 2\beta(2x + \mathrm{i}).$$

Since f has no zeros in $\mathbb{R}$ in x and $\beta \in \mathbb{R}$, and the fast decay at infinity,

$$\frac{f_x}{f} - \frac{\bar{f}_x}{\bar{f}} = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as} \ \ |x| \to \infty,$$

the $\deg(f) = 4$ and there is only one root in $\mathbb{C}_+$, according to the argument principle we have

$$\begin{aligned} Q(u_{\mathrm{double}}, v_{\mathrm{double}}) &= \lim_{R\to\infty} \int_{[-R,R]\cup C_R^+} (|u|^2 + |v|^2) dz = 2\mathrm{i} \lim_{R\to\infty} \int_{[-R,R]\cup C_R^+} \left(\frac{f_x}{f} - \frac{\bar{f}_x}{\bar{f}}\right) dz \\ &= 4\pi(N_{\bar{f}} - N_f) = 8\pi. \end{aligned}$$



(a) Figure 2 with $\varepsilon = 0.5$ together with the straight lines $x + c_1 t = 0$ and $x + c_2 t = 0$

(b) Figure 3 with $\beta = 0$ together with the parabolas $x^2 = \pm\sqrt{3}t$

Figure 4

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Double-Wronskian solutions

In this section, we want to derive the general N-th order rational solution by using the previously developed technique on exact solutions with double-Wronskians.
Let $A \in \mathbb{M}^{2N\times 2N}$ be a complex-valued invertible matrix for $N \in \mathbb{N}$. We define two vectors $\phi, \psi \in \mathbb{C}^{2N}$ from solutions of the linear equations

$$\begin{cases} \partial_\xi \phi = iA\phi, \\ \partial_\eta \phi = iA^{-1}\phi, \end{cases} \quad \text{and} \quad \begin{cases} \partial_\xi \psi = -iA\psi, \\ \partial_\eta \psi = -iA^{-1}\psi, \end{cases} \tag{11}$$

and we introduce the factorization matrix $S \in \mathbb{M}^{2N\times 2N}$ by

$$A = -S\bar{S}, \quad \bar{A} = -\bar{S}S \tag{12}$$

and the connection formula between the two vectors $\phi, \psi \in \mathbb{C}^{2N}$ by

$$\psi = S\bar{\phi}. \tag{13}$$

We recall the conventional notations for double-Wronskian determinants of $(2N) \times (2N)$ matrices:

$$\begin{aligned}
|\widehat{N-1}; \widehat{N-1}| &:= |\phi, \phi', \dots, \phi^{(N-1)}, \psi, \psi', \dots, \psi^{(N-1)}|, \\
|\widetilde{N}; \widetilde{N}| &:= |\phi', \phi'', \dots, \phi^{(N)}, \psi', \psi'', \dots, \psi^{(N)}|, \\
|\overline{N+1}; \overline{N+1}| &:= |\phi'', \phi''', \dots, \phi^{(N)}, \psi'', \psi''', \dots, \psi^{(N+1)}|.
\end{aligned}$$

Theorem

Under the assumptions (11), (12), and (13), the following double-Wronskian

$$\begin{cases} f = |\widetilde{N}; \widehat{N-1}|, \\ \bar{f} = C|\widetilde{N}; \widetilde{N}|, \end{cases} \quad \begin{cases} g = |\widehat{N}; \widetilde{N-1}|, \\ \bar{g} = iC|\overline{N}; \widehat{N}|, \end{cases} \quad \begin{cases} h = iC^{-1}|\widehat{N}; \widehat{N-2}| \\ \bar{h} = C\,\bar{C}^{-1}|\widetilde{N-1}; \widehat{N}| \end{cases} \tag{14}$$

represent exact solutions of the bilinear equations with $C := (-i)^N/|S|$.

The Theorem 1 is derived by using the following bilinear equations:

$$\begin{cases} \mathrm{i}D_\xi(g\cdot f)+2h\bar{f}=0,\\ \mathrm{i}D_\eta(h\cdot \bar{f})+2gf=0,\\ \mathrm{i}D_\xi(f\cdot \bar{f})-2h\bar{h}=0,\\ \mathrm{i}D_\eta(\bar{f}\cdot f)-2g\bar{g}=0, \end{cases} \tag{15}$$

which are transformed from the bilinear equations (5) by the characteristic variables

$$\begin{cases} \xi=\frac{t+x}{4},\\ \eta=\frac{t-x}{4}, \end{cases} \quad \Rightarrow \quad \begin{cases} t=2(\xi+\eta),\\ x=2(\xi-\eta), \end{cases} \quad \Rightarrow \quad \begin{cases} \partial_\xi=2(\partial_t+\partial_x),\\ \partial_\eta=2(\partial_t-\partial_x). \end{cases} \tag{16}$$

Hierarchy of rational solutions

Let L be the nilpotent $(2N) \times (2N)$ matrix with the only nonzero entries being ones on the first lower diagonal. For the Nth-order rational solution which corresponds to the N-multiple eigenvalue at $\zeta = i$, we define

$$A = -I + L, \qquad A^{-1} = -I - L - L^2 - \cdots - L^{2N-1}. \tag{17}$$

Vector $\phi \in \mathbb{C}^{2N}$ satisfies the first system in (11), from which we derive the following recurrent equations for components of ϕ:

$$\partial_\xi \phi_j = -\mathrm{i}\phi_j + \mathrm{i}\phi_{j-1}, \quad j = 1, 2, \dots, 2N \tag{18}$$

closed with $\phi_0 \equiv 0$ and

$$\partial_\eta \phi_j = -\mathrm{i}\phi_j - \mathrm{i}\phi_{j-1} - \mathrm{i}\phi_{j-2} - \cdots - \mathrm{i}\phi_1, \quad j = 1, 2, \dots, 2N. \tag{19}$$

The fundamental solution of the recurrent equations (18) and (19) is defined by using the recursive derivatives as in

$$\phi_j = \frac{1}{(j-1)!}\partial_{\zeta^2}^{j-1} e^{\mathrm{i}(\zeta^2\xi+\zeta^{-2}\eta)}|_{\zeta^2=-1}, \quad j=1,2,\ldots,2N. \tag{20}$$

For parameterization of a general rational solution, we take a linear combination of the fundamental solutions (20) with $2N$ complex parameters:

$$\phi_j = \sum_{k=1}^{j} \frac{c_k}{(j-k)!}\partial_{\zeta^2}^{j-k} e^{\mathrm{i}(\zeta^2\xi+\zeta^{-2}\eta)}|_{\zeta^2=-1}, \quad j=1,2,\ldots,2N. \tag{21}$$

Lemma

Let ϕ be defined by (21) with $c_1, c_2, \ldots, c_{2N} \in \mathbb{C}$ and $\psi = S\bar{\phi}$. The double-Wronskian solutions (14) depend on $2N$ arbitrary real parameters.

The representation (21) can be rewritten in the equivalent form

$$\phi_j = \frac{1}{(j-1)!}\partial_{\zeta^2}^{j-1}\left(\sum_{k=1}^{j} a_k(\zeta^2+1)^{k-1}\right)e^{\mathrm{i}(\zeta^2\xi+\zeta^{-2}\eta+\sum_{k=1}^{j} b_k(\zeta^2+1)^{k-1})}\big|_{\zeta^2=-1}, \quad j=1,\ldots,2N, \tag{22}$$

where $a_1, a_2, \ldots, a_{2N}, b_1, b_2, \ldots, b_{2N} \in \mathbb{R}$. Without loss of generality, we can set $a_1 = 1$ because a_1 can be scaled out by choosing $a_j = a_1\tilde{a}_j$ with new $\tilde{a}_j$ for $j = 2, 3, \ldots, 2N$.

Theorem

Let A be given by (17) with ϕ defined by (20) and $\psi = S\bar{\phi}$ with $A = -S\bar{S}$. The double-Wronskian solutions (14) gives the rational solutions of the MTM system (1) in the form:

$$u = \frac{Q_N(x,t)}{\bar{P}_N(x,t)} e^{-it}, \quad v = \frac{R_N(x,t)}{P_N(x,t)} e^{-it}, \tag{23}$$

where P_N is a polynomial of degree N^2 in x and Q_N, R_N are polynomias of degree $N^2 - 1$ in x.

Examples

The exact solutions for $N = 1, 2$ via double-Wronskian are identical to the expressions given by the Hriota methods.
For $N = 3$, we construct the matrices:

$$A := \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad S := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{2} & 1 & 0 & 0 \\ -\frac{5}{128} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{2} & 1 & 0 \\ -\frac{7}{256} & -\frac{5}{128} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{2} & 1 \end{bmatrix},$$

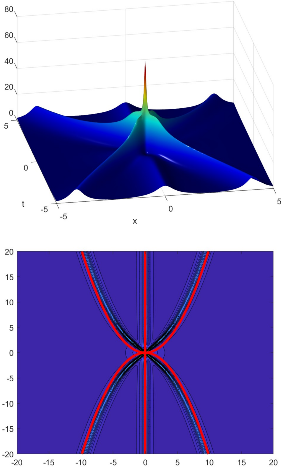
which satisfies $A = -S\bar{S}$.

Using the Wronskian determinant (14) with $C = (-i)^3/|S| = i$, we obtain

$$\begin{cases} f &= |\phi', \phi'', \phi'''; \psi, \psi', \psi''|, \\ g &= |\phi, \phi', \phi'', \phi'''; \psi', \psi''|, \\ h &= -|\phi, \phi', \phi'', \phi'''; \psi, \psi'|, \end{cases}$$

which yield the exact triple algebraic soliton solution. For $c_1 = 1$ and $c_j = 0$ for $j = 2, \ldots, 6$, the exact solution is given by

$$u = \frac{g(x,t)}{\bar{f}(x,t)}, \; v = \frac{h(x,t)}{f(x,t)} \tag{24}$$



Solution surface for $|u|^2 + |v|^2$ versus (x, t) for the solution (24) (top). Contour plot for the 3-soliton solution surface together with $x^2 = \sqrt{9 + 6\sqrt{6}}|t|$ and $x = 0$ (bottom).

The explicit expressions for the parabolas is given by the leading-order terms in the denominator of (24), which are

$$512x^9 - 9216t^2x^5 - 69120t^4x.$$

Making this equal to zero yields three solutions: $x = 0$, $x^4 = (9 + 6\sqrt{6})t^2$, $x^4 = (9 - 6\sqrt{6})t^2$, where the last solution only gives complex values of x, whereas the second solution yields $x^2 = \sqrt{9 + \sqrt{6}}|t|$.

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Conclusion

Summary:

- Constructed the second-order rational solution by using Hirota method.
- Constructed the Nth-order rational solutions by using the double-Wronskian determinants.

Open questions:

- The proof for orbital stability of algebraic solitons.
- The existence of a similar algebraic double-soliton and a similar hierarchy of higher-order rational solutions in the other nonlinear equations associated with the KN spectral problem.
- Development of the IST methods and the Darboux transformation methods for the algebraic solitons associated with the embedded eigenvalues.

Thank You!